Niccolò Guicciardini

ISAAC NEWTON

ON MATHEMATICAL CERTAINTY AND METHOD

Isaac Newton on Mathematical Certainty and Method

Transformations: Studies in the History of Science and Technology

Jed Z. Buchwald, general editor

Dolores L. Augustine, *Red Prometheus: Engineering and Dictatorship in East Germany, 1945–1990*
Lawrence Badash, *A Nuclear Winter's Tale: Science and Politics in the 1980s*
Mordechai Feingold, editor, *Jesuit Science and the Republic of Letters*
Larrie D. Ferreiro, *Ships and Science: The Birth of Naval Architecture in the Scientific Revolution, 1600–1800*
Sander Gliboff, *H.G. Bronn, Ernst Haeckel, and the Origins of German Darwinism: A Study in Translation and Transformation*
Niccolò Guicciardini, *Isaac Newton on Mathematical Certainty and Method*
Kristine Harper, *Weather by the Numbers: The Genesis of Modern Meteorology*
Sungook Hong, *Wireless: From Marconi's Black-Box to the Audion*
Jeff Horn, *The Path Not Taken: French Industrialization in the Age of Revolution, 1750–1830*
Myles W. Jackson, *Harmonious Triads: Physicists, Musicians, and Instrument Makers in Nineteenth-Century Germany*
Myles W. Jackson, *Spectrum of Belief: Joseph von Fraunhofer and the Craft of Precision Optics*
Mi Gyung Kim, *Affinity, That Elusive Dream: A Genealogy of the Chemical Revolution*
Ursula Klein and Wolfgang Lefèvre, *Materials in Eighteenth-Century Science: A Historical Ontology*
John Krige, *American Hegemony and the Postwar Reconstruction of Science in Europe*
Janis Langins, *Conserving the Enlightenment: French Military Engineering from Vauban to the Revolution*
Wolfgang Lefèvre, editor, *Picturing Machines 1400–1700*
Staffan Müller-Wille and Hans-Jörg Rheinberger, editors, *Heredity Produced: At the Crossroads of Biology, Politics, and Culture, 1500–1870*
William R. Newman and Anthony Grafton, editors, *Secrets of Nature: Astrology and Alchemy in Early Modern Europe*
Gianna Pomata and Nancy G. Siraisi, editors, *Historia: Empiricism and Erudition in Early Modern Europe*
Alan J. Rocke, *Nationalizing Science: Adolphe Wurtz and the Battle for French Chemistry*
George Saliba, *Islamic Science and the Making of the European Renaissance*
Nicolás Wey Gómez, *The Tropics of Empire: Why Columbus Sailed South to the Indies*

Isaac Newton on Mathematical Certainty and Method

Niccolò Guicciardini

The MIT Press
Cambridge, Massachusetts
London, England

For information about special quantity discounts, please email
special_sales@mitpress.mit.edu

This book was set in Computer Modern by Compomat s.r.l., Configni (RI), Italy.

Printed and bound in the United States of America.

Library of Congress Cataloging-in-Publication Data

Guicciardini, Niccolò.

Isaac Newton on mathematical certainty and method / Niccolò Guicciardini.

p. cm. - (Transformations : studies in the history of science and technology) Includes bibliographical references and index.

ISBN 978-0-262-01317-8 (hardcover : alk. paper)

1. Newton, Isaac, Sir, 1642–1727—Knowledge-Mathematics. 2. Mathematical analysis. 3. Mathematics-History. I. Title.

QA29.N4 G85 2009
510-dc22

2008053211

10 9 8 7 6 5 4 3 2 1

Nam Geometriae vis & laus omnis in certitudine rerum, certitudo in demonstrationibus luculenter compositis constabat.

For the force of geometry and its every merit laid in the utter certainty of its matters, and that certainty in its splendidly composed demonstrations.

—Isaac Newton, late 1710s (MP, 8, pp. 452–3)

Contents

Preface

There is no doubt that Isaac Newton is one of the most researched giants of the scientific revolution. It is triviality to say that he was a towering mathematician, comparable to Archimedes and Carl Friedrich Gauss. Historians of mathematics have devoted great attention to his work on algebra, series, fluxions, quadratures, and geometry. Most notably, after the publication of the eight volumes of *Mathematical Papers* (1967–1981), edited by D. T. Whiteside, any interested reader can have access to Newton's multifaceted contributions to mathematics. This book has not been written with the purpose of challenging such a treasure of scholarship and information.[1] Rather, I focus on one aspect of Newton's mathematical work that has so far been overlooked, namely, what one could call Newton's philosophy of mathematics.[2] The basic questions that motivate this book are What was mathematics for Newton? and What did being a mathematician mean to him?

It is well known that Newton aimed at injecting certainty into natural philosophy by deploying mathematical reasoning. It seems probable that he entitled his main work *The Mathematical Principles of Natural Philosophy* (1687) in order to state concisely and openly what constituted the superiority of his approach over the Cartesian *Principles of Philosophy* (1644). The role of this program in Newton's philosophical agenda cannot be overestimated. Little research has been devoted, however, to Newton's views on mathematical certainty and method, views that are obviously relevant to his program, for if mathematics is to endow philosophy with certainty, it must be practiced according to criteria that guarantee the certainty of its methods. But the new algebraic methods that Newton mastered so well, and that are the salient characteristic of seventeenth-century mathematical development, appeared to many far from rigorous. Newton participated in the debate on the certainty of mathematical method and elaborated his own answer. However, his views on mathematical method have attracted scant attention from historians,

[1] So writes Whiteside, the doyen of Newtonian studies, about his edition of Newton's mathematical papers: "Let it be enough that the autograph manuscripts now reproduced are no mere resurrected historical curiosities fit only once more to gather dust in some forgotten corner, but will require the rewriting of more than one page in the historical textbook." MP, 7, pp. vii–viii. I dare to hope that my book will in part answer Whiteside's *desideratum*; it would have been impossible to write without a many-years-long full immersion in his eight green volumes.

[2] This is a term that Newton would not have understood.

whereas we know a great deal about, say, René Descartes' or Gottfried Wilhelm Leibniz's philosophies of mathematics.[3]

In order to implement and divulge his innovative approach to a mathematized natural philosophy, Newton tackled a series of questions that have been overlooked or misunderstood by historians, such as When are mathematical methods endowed with certainty? How can one relate the common and new algebraic analyses of the moderns to the venerated methods of the ancients?[4] When is a geometrical construction exact and elegant? What guarantees the applicability of geometry to mechanics? In tackling these issues Newton mobilized deeply felt convictions concerning his role as a philosopher and as a mathematician. He positioned himself against the probabilism endorsed as a moral value by most of the virtuosi of the Royal Society, such as Robert Hooke and Robert Boyle, but he also held a polemical position against two great giants in the common and new analyses, Descartes and Leibniz. On the other hand, Newton found affinities with the thought of Isaac Barrow, Christiaan Huygens, and more obliquely Thomas Hobbes. He reconstructed the development of mathematics from ancient times so as to depict himself as a legitimate heir of the Greek tradition while distancing himself from the moderns. Newton's antimodernism had important consequences for his policy of publication, for the general outlook of his foundational thought, for the mathematical structure of his *Principia*, and for the negotiations he set afoot in order to trade his mathematics in the milieu of British mathematicians and to affirm the superiority of his method over Leibniz's.

In this book I depart from a tendency that has prevailed in the literature devoted to the history of seventeenth-century mathematical methodology. Generally speaking, historians of mathematics have tended to concentrate on questions related to the definitions of basic terms such as *fluxion*, *infinitesimal*, *limit*, and *moment*, that is, on questions concerning *rigor*. These questions, which relate to the "labyrinth of the continuum," certainly had a great importance for a deep thinker like Leibniz. Not so for Newton, who was more concerned with questions about the legitimacy, elegance, and exactness of procedures for solving geometrical problems, that is, with questions about *method*. Leibniz's concerns appear much closer to modern foundational issues in the philosophy of mathematics. Newton's discourse on method is much more opaque to modern readers trained in contemporary foundational literature. This is a natural consequence of scientific training in an era of mathematical

[3] Just to take a telling example, in Mancosu, *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century* (1996), an informed and thorough study of the philosophy of mathematics in the seventeenth century, Newton's name occurs only three times and in passing references.

[4] The terms common analysis and new analysis referred to Cartesian algebra carried on via polynomial equations and to a more general algorithm where infinitesimals, infinite series, and infinite products could be deployed.

practice shaped by the exigencies that emerged after the early-nineteenth-century rigorization of the calculus, axiomatization of geometry, and development of abstract algebra. Since then, definitions of basic terms have become a prominent feature of the foundations of mathematics. Newton did devote attention to, say, the nature of infinitesimals, but he was much more concerned with methodological issues related to the analysis (or resolution) and synthesis (or composition) of geometrical problems. In this he showed himself to be influenced by Descartes, who was deeply involved in the program of redefining his algebraic method of analysis as an alternative to the analysis of the ancients ("Analysis Veterum").[5] Descartes also traced the boundaries between legitimate (exact) and illegitimate (not exact) means of geometrical construction and classed them on the basis of criteria of simplicity that broke with the ancient tradition conveyed by Pappus's *Collectio*. Newton fiercely rejected Descartes' canon of problem resolution and composition, and proposed an alternative that looked to ancient exemplars as models superior to the mathematics practiced by the "men of recent times." But Newton's admiration for the ancients—arguably intertwined with many facets of his philosophy of nature and religion—opened a gap between his views on mathematical method and his mathematical practice.

Several hitherto unexplained aspects of Newton's mathematical work are related to this condition of stress and strain characterizing his thoughts on mathematical method. Why did Newton fail to print his method of series and fluxions before the inception of the priority dispute with Leibniz? Did he use his new analysis in the *Principia*, as he claimed in retrospect? And if so, why did he hide his competence in his new methods of quadrature when writing the *Principia*? Why does his *Arithmetica Universalis*, a work devoted to algebra, end with pointed criticisms of the use of algebraic criteria in the construction of geometrical problems? Why did Newton, a master of symbolic manipulation, express deep disparagement of the algebraists (whom he, according to hearsay, labeled the "bunglers of mathematics")?[6] Which strategies did he adopt in order to maintain the superiority of his method compared to Leibniz's extraordinarily efficient algorithm? Why did he engineer *Commercium Epistolicum* (1713), which was meant to prove his priority in the discovery of the calculus, using documents that proved everything he had achieved on series and quadrature but revealing nothing about the rules of the calculus? Why did he—*contra* Leibniz—attribute these rules to Isaac Barrow rather than to himself? Was not therefore his policy during the priority dispute lacunose and contradictory?

Historians have found these questions embarrassing. It is not unusual to encounter attempts to formulate answers in terms of psychological motivations, a

[5] Bos, *Redefining Geometrical Exactness* (2001).
[6] Hiscock, *David Gregory, Isaac Newton and Their Circle* (1937), p. 42.

sign—in my view—of the difficulty of making sense of Newton's policy and convictions. So, we often read that it was Newton's neurotic shyness, or a lack of interest in mathematics that suddenly crept into his mind, that prevented him from printing his method. Further, we are told that in the *Principia* he studiously tried to be obscure in order to avoid dispute and criticisms because of his obsessive feeling of being persecuted. We even encounter the thesis according to which, in the polemic against algebra and calculus, he was trying to hide his indebtedness toward an intellectual father (Descartes) whose image he found oppressive. When we move to consider the polemic with Leibniz, we find studies that highlight Newton's obsessive approach to the priority dispute, his lack of fairness, his egotism, and even his political motivations related to the Hanoverian succession. Little effort is made to try to discern in such a muddied context (and muddied and political it certainly was!) coherent methodological positions held by Newton and by his opponents.

I believe we should be able to find better answers by studying more carefully Newton's writings on the nature of mathematics, which are so abundant in his manuscripts. True, the search for Newton the philosopher of mathematics is at times frustrating. What we find is a disconnected, sometimes contradictory, constellation of pronouncements scattered in the margins of mathematical manuscripts, in aborted prefaces and appendices, in letters and personal notes. They serve their purpose in a dialogical context, providing defensive grounds for mathematical practices, orienting aims, and establishing hierarchies. However, in these writings Newton reveals himself as a mathematician who—even when shattered by psychological disturbances, stymied by academic rivalries, and motivated by political interests—is able to endorse a fairly clear view of method and certainty. It is this view that prompted him to articulate his mathematical work according to codes of communication that were understood by his contemporaries, especially by his close acolytes, but that turn out to be so puzzling for modern readers. In short, even in the heated context of the priority dispute Newton has something to tell us about mathematical method. We should consider his theses, even though his stature as a philosopher of mathematics is inferior to a Leibniz or a Descartes.

In this book I study Newton's methodology of mathematics by analyzing his main works, from the early treatises on series and fluxions to the writings addressed against Leibniz. Even though my book is not devoted to Newton's mathematical results, it is important to try to understand his mathematics, because his views on mathematical method interacted with his mathematical practice in a complex way. Therefore, the reader will find some pages in which I analyze Newton's mathematics at work. I have been extremely selective, and the examples I have chosen are not meant to offer an exhaustive view of Newton's mathematics; he was simply too prolific to allow condensing the wealth of his results in a single book. I have tried, however, to choose examples that are simple enough to be followed by a reader equipped with a modicum of mathematical expertise and that are representative of a method or an approach that Newton developed. The expert mathematician will find most of the examples far too simple. Newton was indeed a great problem solver, and his best mathematics can be admired by reading *Mathematical Papers* and its editor's profound commentary.

In matters of notation I have avoided translating Newton's mathematics into modern terms.[7] The mathematician not trained in history will find Newton's mathematical language and practice somewhat puzzling. For instance, Newton and his contemporaries did not talk about functions, but about continuously varying magnitudes. Newton did not use an equivalent of Leibniz's integral sign consistently; most often he used connected prose and referred to "the area under the curve" or "the fluent of." He was also somewhat confused about the distinction between definite and indefinite integral, and never rendered constants of integration explicit. Therefore, he often used the singular ("the fluent of"), but he was of course aware that the indefinite integral identifies a class of functions. Further, he dealt with the convergence of infinite series in very intuitive terms; their convergence was tested in cavalier terms. We have to wait for Augustin Louis Cauchy at the beginning of the nineteenth century for a modern theory of convergence. The list of oddities could continue. I have made no effort (with the exception of a few explanatory footnotes) to avoid the distinctive character of Newton's mathematics compared to modern usage.

The book is divided into six parts. Part I provides some preliminaries: a survey of Newton's mathematical work and of the development of his ideas on mathematical method that began to mature just after the creative burst of the *anni mirabiles* (chapter 1); a comparison between his youthful program in natural philosophy with the one endorsed by influential contemporaries like Descartes, Hooke, and Boyle (chapter 2); a presentation of Descartes' ideas on analysis and synthesis as Newton found them in the *Géométrie* (chapter 3). In fact, Newton, who was in his mathematical practice so much a Cartesian, stood in opposition to Descartes with respect to method.

Part II considers the first period of Newton's methodological thought. He began distancing himself from Cartesian method in writings that date from the 1670s in which he compared common analysis (i.e., Cartesian algebra) to ancient analysis. The occasion for these reflections was Newton's involvement in the project of revising a textbook on algebra by the Dutchman Gerard Kinckhuysen and his commitment to prepare lectures on algebra (chapter 4). It is in this context that Newton began reading Pappus with the purpose of recovering the ancient analysis (chapter 5). In this period Newton also worked on cubic curves (chapter 6); and it is in this research that the tensions between his mathematical practice and his views on method surfaced.

---

[7] So, for instance, I avoid the integral sign and prefer to talk about the "quadrature of a curve" rather than the "integral of a function."

Part III is devoted to Newton's attempts to provide a synthetic version of what he labeled "method of series and fluxions." Newton's early researches on series (chapter 7) and fluxions (chapter 8) were carried on in terms unacceptable to his more mature standards of validation. Consequently, he aimed at developing a synthetic method of fluxions—a "more natural approach," as he says—consonant with the practice of the ancient geometers. These researches culminated in the early 1680s in a treatise entitled "Geometria Curvilinea," in which Newton elaborated his method of first and ultimate ratios, a method that informs the most mature presentation of his method of series and fluxions offered in the *Principia* and *De Quadratura* (chapter 9).

Part IV considers the mathematical methods employed by Newton in the *Principia* (chapter 10). I devote particular attention to the strategies he chose in order to accommodate his analytical methods in common analysis (chapter 11) and new analysis (chapter 12) in the body of a text that he presented as written according to the "ancient and good geometry."

Part V concerns perhaps the most philosophically freighted texts that Newton wrote in the 1690s and early 1710s. In these decades his belief in the myth about a *prisca sapientia* of the ancients prospered and determined his self-portraiture as an heir to the mathematics of Euclid and Apollonius. Consequently, he wrote at length on the relations among analysis, synthesis, algebra, natural philosophy, and mechanics (chapters 13 and 14).

Newton's views concerning mathematical method emerge again in the polemic with Leibniz, which occupied him especially from 1710 (chapter 15). In order to trace the rationale for Newton's polemical strategy, Part VI devotes attention to his policy of publication adopted in circulating manuscripts, in correspondence (chapter 16), and later in printing some of his tracts on the new analysis that he had composed years before (chapter 17).

Acknowledgments

During the long period necessary to write this book I profited from the advice of a great number of scholars. First, I should mention my debt to two giants in the Newtonian scholarship, I. Bernard Cohen and Derek T. Whiteside; without their impressive works and the generous support they gave me through correspondence, my research would not have been possible. Colleagues in Siena (Emanuela Scribano and Alessandro Linguiti), Milan (Umberto Bottazzini), and Bergamo (Andrea Bottani, Richard Davies, and Maddalena Bonelli) were of help in many ways (thank you, Ric, for rescuing me from the pitfalls of too many "Italianisms").

It is a pleasure also to thank Fabio Acerbi, Franco Giudice, Massimo Mugnai, and Michael Nauenberg for much-needed assistance on ancient geometry, optics, Leibniz, and mathematical physics, respectively. A one-year sabbatical spent as a Visiting Fellow at Clare Hall (Cambridge) allowed me to begin planning my book in a friendly

atmosphere. At Cambridge I exploited Scott Mandelbrote's kind willingness to share his expertise in Newtoniana with me.

A subsequent semester as Mellon Visiting Professor at the California Institute of Technology gave me the chance of working in one of the most important centers for early-modern history of science and Newtonianism in particular. I thank the Andrew W. Mellon Foundation for generous support. At Caltech the exchanges I had with Jed Buchwald, André Carus, Moti Feingold, Kristine Haugen, Diana Kormos-Buchwald, Rose-Mary Sargent at Dabney and with Tinne Hoff Kjeldsen, Andrea Loettgers, Jesper Lützen, Tilman Sauer during seminars at the Einstein Papers Project, and a memorable visit from George Smith and Christopher Smeenk, shaped many of the theses defended in this book. During meetings of the Wissenschaftlicher Beirat of the Bernoulli-Edition in Basel I received advice from one of the greatest Leibniz scholars, Eberhard Knobloch.

My former supervisor and present friend Ivor Grattan-Guinness and the distinguished physicist and writer Klaas Landsman read and extensively commented on a draft of this book. Henk Bos's detailed comments rescued me from a great number of blunders; I am grateful to him for the extreme, and undeserved, kindness with which he addressed the task of revising my script, equaled by the widely appreciated rigor of his scholarship. In the very last stage of preparation of the book for the press, Massimo Galuzzi rendered his profound knowledge of Cartesian and Newtonian mathematics available to me with a generosity exemplary of what excellency in scholarship should be; I consider it great luck to have him as one of my correspondents and friends.

The competence of the librarians working in the "Angelo Mai" library in Bergamo, the library of the Institute and Museum of the History of Science in Florence, the University Library in Cambridge, the Royal Society Library in London, the Huntington Library in San Marino, and the Millikan Library at Caltech rendered this book possible. Sara Damiani of the Centro Arti Visive of the University of Bergamo generously helped me in preparing the figures with Photoshop. Silvia Maschio of Compomat s.r.l transformed my sloppy LaTeX file into a magnificent camera-ready pdf. The help of Laura M. Keith and Sergio Knipe in revising the English is gratefully acknowledged. Without the patience and kindness of the staff at MIT Press, producing this book would have been much more difficult.

The greatest "thank you" is for my wife, Elena, and my daughter, Pepi; they know why.

Abbreviations and Conventions

All dates are old style (O.S.) unless otherwise specified (N.S.). When necessary, I use January 17/27, 1712/13 for January 17, 1712 (O.S.) = January 27, 1713 (N.S.).

The following abbreviations are employed:

§ refers to section numbers in this book.

$\propto$ = "is proportional to."

"Account" = "An Account of the Book Entituled Commercium Epistolicum Collinii & Aliorum de Analysi Promota; Published by Order of the Royal Society, in Relation to the Dispute Between Mr. Leibnitz and Dr. Keill, About the Right Invention of the Method of Fluxions, by Some Call'd the Differential Method." *Philosophical Transactions* 29, no. 342 (1715): 173–224. Author: Isaac Newton. Published anonymously. Facsimile in Hall, *Philosophers at War* (1980), pp. 263–314.

Add. = Additional manuscript in the Cambridge University Library.

AT = Descartes, René. *Oeuvres de Descartes.* Edited by C. Adam and P. Tannery. 11 vols. New ed. Paris: Vrin, 1964–67.

Commercium Epistolicum = *Commercium Epistolicum D. Johannis Collins, et Aliorum De Analysi Promota: Jussu Societatis Regiae In Lucem Editum.* London: typis Pearsonianis, 1712 (printed late 1712, distributed January–February 1713 (N.S.)).

Correspondence = Newton, Isaac. *The Correspondence of Isaac Newton.* Edited by H. W. Turnbull, J. F. Scott, A. R. Hall, and L. Tilling. 7 vols. Cambridge: Cambridge University Press, 1959–77.

De Analysi = MS LXXXI, no. 2 (Royal Society of London). Author: Isaac Newton. Title: *De Analysi per Aequationes Numero Terminorum Infinitas.* Date of composition: 1669. Edited in MP, 2, pp. 206–47. First printed in Newton, *Analysis per Quantitatum* (1711), pp. 1–21.

De Methodis = MS Add. 3960.14 (Cambridge University Library). Author: Isaac Newton. Title: untitled since the first folio is lacking, known as *Tractatus de Methodis Serierum et Fluxionum.* Date of composition: 1670–1671. Edited in MP, 3, pp. 38–328 (on pp. 32–7 the first missing leaf is from a transcript by William Jones). First printed in English translation as Newton, *Method of Fluxions* (1736).

De Quadratura = *Tractatus de Quadratura Curvarum.* First printed in Newton, *Opticks* (1704), pp. 165–211. Excerpts were communicated to Wallis for inclusion in volume 2 of his *Opera* (1693), pp. 390–6. Date of composition: early versions in 1691–1692, revised for publication in 1703. The numerous early versions and the revisions (mostly in MSS Add. 3960.7–13, 3962.1–3, and 3965.6 (Cambridge University Library)) are edited in MP, 7, pp. 24–182, and MP, 8, pp. 92–167.

Enumeratio = MS Add. 3961.2, ff.1r–14r (Cambridge University Library). Author: Isaac Newton. Title: *Enumeratio Linearum Tertii Ordinis*. Date of composition: 1695. Edited in MP, 7, pp. 588–645. First printed in Newton, *Opticks* (1704), pp. 139–62 (+ 6 Tables).

Geometria = Descartes, René. *Geometria, à Renato Des Cartes Anno 1637 Gallicè Edita*. Amsterdam: Apud Ludovicum & Danielem Elzevirios, 1659–61.

"Geometriae Libri Duo" = Edited by D. T. Whiteside mostly from Add. 3962.1, 3962.3, 3963.2, and 4004 (Cambridge University Library). Author: Isaac Newton. Title: "Geometriae Liber Primus" and "Geometriae Liber Secundus." Date of composition: mid-1690s. Edited in MP, 7, pp. 402–561.

Géométrie = Descartes, René. *La Géométrie*. In René Descartes, *Discours de la Méthode, pour Bien Conduire sa Raison et Chercher la Verité dans les Sciences: Plus la Dioptrique, Les Meteores, et la Geometrie, Qui Sont des Essais de Cete Methode*, pp. 297–413. Leiden: Maire, 1637.

Intended Preface = MS Add. 3968.9, f. 109 (Cambridge University Library). Author: Isaac Newton. Date of composition: late 1710s. Edited in MP, 8, pp. 452–59. Preliminary drafts in MP, 8, pp. 442–52. Cohen and Whitman provide an English translation in Newton, *Principles*, pp. 49–54.

Lucasian Lectures on Algebra = MS Dd.9.68 (Cambridge University Library). Author: Isaac Newton. Title: untitled. Edited in MP, 5, pp. 54–491. Deposited late 1683 or early 1684. First printed as Newton, *Arithmetica Universalis* (1707).

MP = Newton, Isaac. *The Mathematical Papers of Isaac Newton*. Edited by D. T. Whiteside. 8 vols. Cambridge: Cambridge University Press, 1967–81.

Optical Lectures = MS Add. 4002 (Cambridge University Library). Author: Isaac Newton. Title: *Lectiones Opticae*. Date of composition: 1669/70—2. Edited in Newton, *The Optical Papers* (1984), pp. 46–279. A later version, MS Dd.9.67 (Cambridge University Library), entitled *Opticae pars 1ª* and *Opticae pars 2ᵈᵃ*, deposited in October 1674, appeared posthumously, first in an English translation of *pars 1ª* as *Optical Lectures: Read in the Publick Schools of the University of Cambridge* (1728), and then the complete Latin text as *Lectiones Opticae* (1729).

Preface = "Praefatio ad Lectorem." In Newton, Isaac. *Philosophiae Naturalis Principia Mathematica* (1687), pp. i–ii. Facsimile (as in third (1726) edition) in *Principia*, pp. 15–7. English translation in *Principles*, pp. 381–3.

Principia = Newton, Isaac. *Philosophiae Naturalis Principia Mathematica, The Third Edition (1726) With Variant Readings*. Edited by A. Koyré and I. B. Cohen, assisted by A. Whitman. Cambridge: Cambridge University Press, 1972.

Principles = Newton, Isaac. *The Principia: Mathematical Principles of Natural Philosophy . . . Preceded by a Guide to Newton's Principia by I. Bernard Cohen.* Translated by I. B. Cohen and A. Whitman, assisted by J. Budenz. Berkeley: University of California Press, 1999.

Isaac Newton on Mathematical Certainty and Method

I Preliminaries

It is useful to have a general scheme of Newton's mathematical work at the outset, before delving into the details of his works on algebra (part II), fluxions (part III), natural philosophy (part IV), geometry (part V), and the more philosophical issues that emerged in the context of the polemic with Leibniz (part VI). Therefore, in chapter 1, I give a general overview of the development of Newton's ideas on mathematical method, from his *anni mirabiles* down to his maturity.

Chapter 2 considers a program that Newton stated as a very young Lucasian Professor in his *Optical Lectures.* According to this program, which never ceased to inform Newton's agenda, mathematics is the vehicle that can guarantee certainty in natural philosophy. This—he was adamant—had escaped Descartes and his followers, who had remained trapped in a qualitative and hypothetical mode of explanation of natural phenomena. When considering the ideas that Newton later developed concerning mathematical method, this agenda should always be kept in mind.

Chapter 3 considers the great antagonist against whom Newton positioned himself: Descartes. What Descartes had to say in the *Géométrie* (1637) concerning method and certainty was rejected by Newton for reasons which become the object of study for the rest of the book. We cannot understand Newton's methodology unless we are aware of the antagonist that he always had in mind, and appreciate that his polemic against Cartesian method engendered a deep tension between his mathematical practice, which was indebted to Cartesian ideas, and his methodological views, which were at odds with it. Therefore, chapter 3 considers the basic elements of the Cartesian mathematical canon.

1 Newton on Mathematical Method: A Survey

For in those days I was in the prime of my age for invention & minded Mathematicks & Philosophy more then at any time since.

— Isaac Newton, 1718

1.1 Early Influences

Mathematics played a prominent role in Newton's intellectual career. This was not, of course, his only concern. A polymath and polyhistor, Newton devoted years of intense research to the reading of the Books of Nature and Scripture, deploying the tools of the accomplished "chymist" (at the furnace and at the desk), instrument maker (he made his own instruments, among them the first reflecting telescope), experimentalist, astronomer, biblical interpreter, and chronologist. In all these fields mathematics entered as one of the most powerful and reliable tools for prediction and problem solving, and as the language that guaranteed accuracy and certainty of deduction. Newton would not have achieved most of his results without it.[1] It is no coincidence that the adjective mathematical enters into the title of his masterpiece.

When Newton matriculated at Cambridge in 1661, he possessed only a modicum of mathematical training. Two years later the first Lucasian Chair of Mathematics was conferred on Isaac Barrow, a scholar of broad culture who would play an important role in Newton's intellectual life. The existence of such chairs, which provided mathematical teaching at the universities, was something of a novelty in England.[2] Barrow therefore had to defend his discipline and lectured on the usefulness of mathematical learning. He did so in verbose and scholarly lectures, which Newton probably attended. Barrow patterned his peroration following the agenda set by Proclus, and he had in mind a late-sixteenth-century debate over the certainty of mathematics, which was sparked in 1547 by Alessandro Piccolomini's commentary

---

Epigraph from MS Add. 3968.41, f. 85r. For a discussion of this memorandum see Westfall, *Never at Rest* (1980), p. 143, and "Newton's Marvelous Years of Discovery and Their Aftermath" (1980); Hall, *Philosophers at War* (1980), pp. 10–23. See also Whiteside, "Newton's Marvellous Year" (1966). The best guide to Newton's mathematical work is to be found in Whiteside's commentary to *Mathematical Papers*.

[1] Jed Buchwald and Mordechai Feingold are currently examining Newton's work on chronology. Their research reveals the importance of new mathematical techniques in treating astronomical and historical data.

[2] For the antecedents, see Feingold, *The Mathematicians' Apprenticeship* (1984).

Table 1.1 Mathematical Books Annotated by Newton in the 1660s

René Descartes	*Geometria, à Renato des Cartes*, Amsterdam, 1659–61
François Viète	*Opera Mathematica*, Leiden, 1646
Frans van Schooten	*Exercitationum Mathematicarum*, Leiden, 1657
William Oughtred	*Clavis Mathematicae*, 3d ed., Oxford, 1652
John Wallis	*Operum Mathematicorum Pars Altera*, Oxford, 1656
John Wallis	*Commercium Epistolicum*, Oxford, 1658

on pseudo-Aristotle's *Problemata Mechanica*.[3] The rising status of mathematics was opposed by some Aristotelian philosophers like Piccolomini, who maintained that mathematics did not possess the deductive purity of syllogistic logic and was not a science because it did not reveal causal relationships. Barrow's defense of geometry as a model of reasoning and his idea that since geometrical magnitudes are generated by motion, a causal relationship can be captured in such mechanically based geometry must have impressed the young scholar. These typically Barrovian ideas remained the backbone of Newton's views about mathematics.

Newton soon began to read advanced mathematical texts, possibly borrowing them from the Lucasian Professor. The mathematical books he had on his desk, which he annotated extensively, are listed in table 1.1. As is often repeated in later memoranda and hagiographic biographies, he devoted little attention to ancient geometry, which is at odds with his mature predilection for the ancients, which began to flourish in the 1670s. As far as we know, of the ancient corpus he studied only Euclid's *Elements* in Barrow's algebraized edition.[4] He learned algebraic notation from Oughtred's *Clavis Mathematicae* in the third 1652 edition, and from Viète's *Opera Mathematica* (1646). These last two works were based on the idea that algebra is not a deductive theory, like the *Elements*, but rather an analytical, heuristic tool that can extend the possibility of finding solutions to problems, especially geometrical problems. The annotations to Oughtred and Viète show how interested Newton was in this promising method of discovery.[5] Algebra was still a novel language in England. Oughtred had been a pioneer (his *Clavis* had first appeared in 1631), but in the 1660s there was still need for an updated text on algebra. In 1669, Newton became involved in the project of producing such a text-

[3] Piccolomini, *In Mechanicas Quaestiones Aristotelis* (1547). On Barrow's reading of Proclus, see Stewart, "Mathematics as Philosophy: Proclus and Barrow" (2000). On the debate initiated by Piccolomini, there is a vast literature; see Jardine, "Keeping Order in the School of Padua" (1997).
[4] *Euclidis Elementorum Libri XV Breviter Demonstrati* (1655).
[5] MP, 1, pp. 25–88.

book (see chapter 4). Algebra was interesting as a tool for practical applications (it answered the needs of cartographers, instrument makers, mechanics, accountants, land surveyors) but it was also promising for more theoretical purposes. The latter motivation was the stimulus for Newton.

The seminal text in Newton's mathematical formation is a highly abstract essay: Descartes' *Géométrie*. He borrowed and annotated the second Latin edition (1659–1661) by Frans van Schooten.[6] Here Descartes had proposed a novel method for the solution—he claimed in the opening sentence—of all the problems of geometry. It was on this text that Newton concentrated his attention. Descartes taught how geometrical problems could be expressed in terms of algebraic equations (this process was termed the resolution or analysis of the problem). He maintained that finding the equation and determining its roots, either by finite formulas or approximations, is not the solution of the problem (see chapter 3). It was not a surprise for the contemporaries of Descartes and Newton to read that in order to reach the solution, one had to geometrically construct the required geometrical object. A geometrical problem called for a geometrical construction (a composition or synthesis), not an algebraic result. Traditionally, such constructions were carried out by means of intersecting curves. Thus, Descartes provided prescriptions to construct segments that geometrically represent the roots and are therefore the solution of the problem.

By Newton's day the heuristic method proposed by Descartes was labeled common analysis. It was contrasted with a more powerful new analysis, which tackled problems about tangents and curvature of curves and about the determination of areas and volumes that cannot be reached by the finitist means envisaged by Descartes. Common analysis proceeds by "finite" equations (algebraic equations, we would say) in which the symbols are combined by a finite number of elementary operations. The new analysis instead goes beyond these limitations because it makes use of the infinite and infinitesimal.

Basically, Newton and his contemporaries understood both the common analysis and the new analysis, where respectively "finite" and "infinite equations" (infinite series and infinite products) were deployed, as heuristic tools useful in discovering a solution. Analysis, however, had to be followed by synthesis, which alone, in their opinion, could provide a certain demonstration. Barrow much concerned himself with synthesis and, in his lectures defending mathematical certainty, aimed to provide synthetic demonstrations of the results reached by the heuristic techniques characteristic of the new analysis. His young protégé was making inventive forays into the new analysis. Newton was aware, however, that a synthetic construction was needed, and he later turned to Barrow for inspiration.

[6] Newton worked on the second Latin edition, but he might also have encountered the smaller first Latin edition prepared by van Schooten, which appeared in 1649. A copy of the first edition (University Library (Cambridge) Adv.d.39.1) might have been in Newton's possession, but "its brief manuscript annotations are not in Newton's hand." MP, 1, p. 21.

1.2 First Steps

Newton's early notes on Descartes' *Géométrie* reveal how quick he was in mastering algebra applied to geometry. In 1665 he began to think about how the equation could reveal properties of the curve associated to it via a coordinate system. Actually, he began to experiment with alternative coordinate systems to orthogonal or oblique axes. He tried, for instance, what we call polar, bipolar, or pedal coordinates. He also began to work with transformation of coordinates. One line of research consisted in trying to extend algebraic treatment beyond the conic sections. In *De Sectionibus Conicis, Nova Methodo Expositis Tractatus*, which Newton read in the *Operum Mathematicorum Pars Altera* (1656), Wallis had developed an algebraic treatment of conics as graphs of second-degree equations in two unknowns. Newton began to extend the definitions of diameter, chord, axis, vertex, center, and asymptote to higher-order algebraic curves. In the late 1660s he made his first attempts to graph and classify cubic curves.[7]

Another line of research concerned the so-called organic description (or generation) of curves.[8] This was an important topic, since in order to determine the point of intersection of curves in the construction of geometrical solutions, it was natural to think of the curves as generated by a continuous motion driven by some instrument (an $o\rho\gamma\alpha\nu o\nu$). It is the continuity of the motion generating the curves that guarantees a point of intersection can be located exactly. Descartes had devised several mechanisms for generating curves. In *De Organica Conicarum Sectionum in Plano Descriptione Tractatus* (1646), which Newton read in *Exercitationum Mathematicarum* (1657), van Schooten had presented several mechanisms for generating conic sections. This research field was connected with practical applications, for instance, lens grinding and sundial design, but it was also sanctioned by classical tradition and motivated the highly abstract needs underlined by Descartes. Newton was able to devise a mechanism for generating conics and to extend it to higher-order curves (§5.4).

In 1665, Newton deployed organic descriptions in order to determine tangents to mechanical lines, that is, plane curves such as the spiral, the cycloid, and the quadratrix that Descartes had banned from his *Géométrie* (see chapter 3). The study of mechanical lines, curves that do not have an algebraic defining equation, was indeed a new, important research field. How to deal with them was unclear. Newton was able to determine the tangent to any curve generated by some tracing mechanism. He decomposed the motion of the tracing point P, which generates the curve, into two components and applied the parallelogram law to the instantaneous component velocities of P (see the parallelogram law on the top of the left margin

---

[7] MP, 1, pp. 155–244.
[8] MP, 2, pp. 134–42 and 152–5.

in figure 1.1). For instance, the point of intersection of two moving curves will generate a new curve whose tangent Newton was able to determine. Such a method for determining tangents without calculation pleased Newton as much as did his new techniques for the organic description of conics. This was an approach to the study of curves—alternative to the Cartesian algebraic—that Barrow had promoted and that in the 1670s Newton began to couple with ideas in projective geometry. Already in 1665 the master in the common and new algebraic analyses was experimenting with non algebraic approaches to geometrical problems.

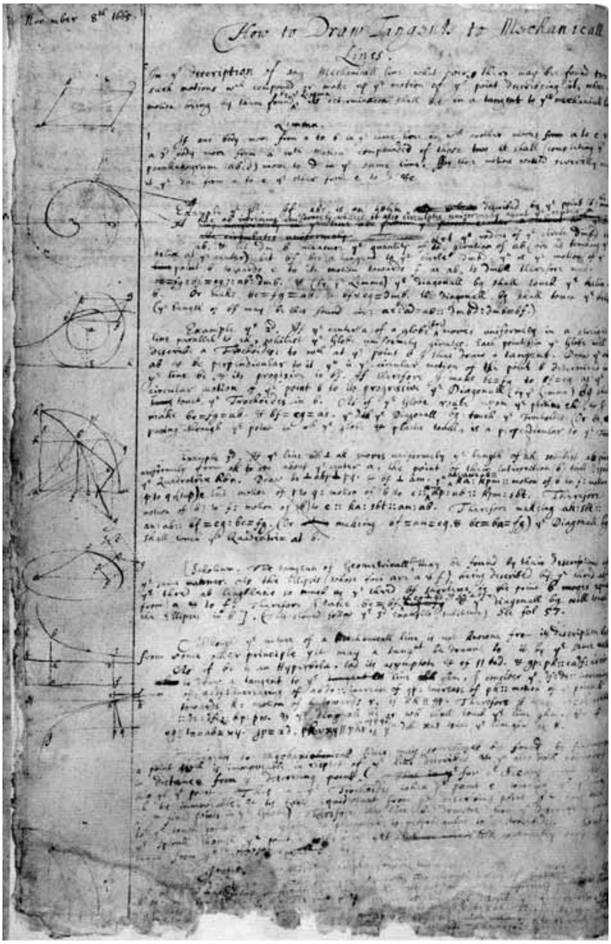
In these early researches one encounters a characteristic of Newton's mathematical practice, a deep intertwining between algebra and geometry, that eventually led to unresolved tensions in his views on mathematical certainty and method. Indeed, it is often the case that in tackling a problem Newton made recourse to a baroque repertoire of methods: one encounters in the same folios algebraic equations, geometrical infinitesimals, infinite series, diagrams constructed according to Euclidean techniques, insights in projective geometry, quadratures techniques equivalent to sophisticated integrations, curves traced via mechanical instruments, numerical approximations. Newton's mathematical toolbox was rich and fragmentary; its owner mastered every instrument it contained with versatility. But he was also a natural philosopher who envisaged a role for mathematics that did not allow him to leave the toolbox messy, albeit efficient, and open for unauthorized inspection.

1.3 Plane Curves

How did the young Newton tackle a problem that was quite difficult in his day: the drawing of tangents to plane curves? Figure 1.1 shows the first folio of a manuscript dated by Newton (in retrospect?) November 8th, 1665, and entitled "How to Draw Tangents to Mechanicall Lines." In the left margin there are an Archimedean spiral, a trochoid, and a quadratrix.[9]

Tracing the tangent to the spiral was particularly handy. To a point b of a spiral with pole a (see figure 1.2) Newton associated a parallelogram having a vertex in b whose sides, the former bc directed along the radius vector ab and the latter bf orthogonal to it, are proportional to the radial speed and to the transverse speed of b. The diagonal bg determines the tangent at b. In other cases, the method was more difficult to implement, and Newton made a couple of blunders, which he soon corrected, in tracing the tangent to the quadratrix and to the ellipse.[10]

[9] In modern symbols these three curves have equations $r = c_0\theta$ (r, θ polar coordinates, c_0 constant), $x = c_1 t - c_2 \sin t$, $y = c_1 - c_2 \cos t$ (parametric equations, $c_2 < c_1$), and $x = y \cot(\pi y/2c_3)$ (x, y, Cartesian coordinates).
[10] MP, 1, pp. 379–80.


November 8th 1665.

How to Draw Tangents to Mechanicall Lines.

Figure 1.1

Newton's kinematic method for drawing tangents to mechanical curves. From top to bottom of the left margin, below the illustration of the parallelogram, are the following curves. (i) The Archimedean spiral is traced by a point that slides with constant speed along a straight line that rotates with constant angular speed. (ii) The trochoid (sometimes called curtate cycloid) is traced by a point on a disk that rolls without sliding along a straight line. (iii) The quadratrix is a curve traced by the intersection of a radius and a line segment moving at corresponding rates. A square and a circle are drawn so that one corner of the square is the center of the circle, and the side of the square is the radius of the circle. A radius rotates clockwise from the side of the square to the base at a constant angular speed. At the same time, a line segment falls from the top of the square at constant vertical speed and remains parallel to the base of the square. Both start moving at the same time, and both hit the bottom at the same time. Newton also considers two "Geometricall lines," namely (iv) the ellipse, and (v) the hyperbola. Source: Add. 4004, f. 50v. Reproduced by kind permission of the Syndics of Cambridge University Library.

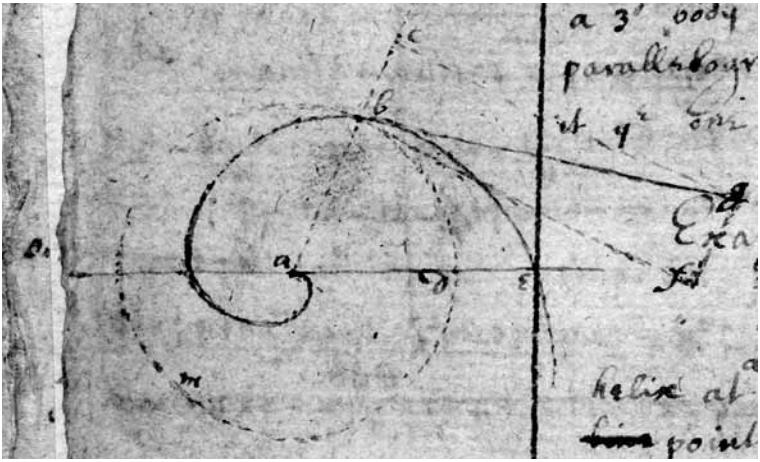


Figure 1.2
Tracing the tangent to the Archimedean spiral. Source: Add. 4004, f. 50v. Reproduced by kind permission of the Syndics of Cambridge University Library.

Newton applied his method for drawing tangents not only to mechanical but also to geometrical lines: the ellipse and the hyperbola.[11]

In his early papers, Newton intertwined the geometrical approach to tangents with the development of a new algorithm, which he called the method of series and fluxions. This method allowed the calculation of the tangent and curvature to all plane curves known in Newton's day. Later, I describe Newton's algorithm for the determination of tangents (§8.3.6) and its application to the conchoid (Cartesian equation $x^2y^2 = (c_1 + y)^2(c_2^2 - y^2)$). Undoubtedly, this algorithm, referred to in modern textbooks as the calculus, is the most celebrated discovery that Newton made in the years 1664–1666. This highly symbolic and algebraized tool of problem solving is discussed in part III. It should be stressed, however, that what appears, with the benefit of hindsight, to be Newton's greatest achievement was perceived as just one among many alternative approaches to problem solving by its inventor.

Infinite series allowed Newton to study the properties of mechanical curves, such as the cycloid (the curve traced by a point on the circumference of a circle that rolls along a straight line: the parametric equation of the cycloid generated by a circle with radius a is $x = a(t - \sin t)$, $y = a(1 - \cos t)$).[12] Most notably, they

[11] MP, 1, pp. 369–99. See also the beginning of the "October 1666 Tract on Fluxions." MP, 1, pp. 400–1. Kirsti Andersen studied this technique and presented her analysis at a meeting in Oberwolfach (Germany) in December 2005; see Andersen, "Newton's Inventive Use of Kinematics in Developing His Method of Fluxions" (2005).
[12] As Newton wrote in 1684, "To be sure, convergent equations can be found for the curved lines

allowed him to calculate curvilinear areas, curvilinear volumes, and arc lengths; these calculations were generally called "quadrature problems." So, squaring a curve meant calculating the area of the surface bounded by it. Nowadays we would use Leibnizian terminology and speak about problems in integration. Sections §7.4 and in §8.4.5 take up Newton's calculation of the area of the surface subtended by the cycloid and by the cissoid (Cartesian equation $y^2(a-x) = x^3$). The fact that Newton's method allowed him to tackle mechanical curves and quadratures is due to a mathematical fact of which he was well aware. Using Leibnizian jargon, we can say that while differentiation of algebraic functions (accepted by Descartes) leads to algebraic functions, integration can lead to new transcendental functions. Newton referred to what are now called transcendental functions as quantities "which cannot be determined and expressed by any geometrical technique, such as the areas and lengths of curves."[13] Infinite power series—in some cases fractional power series—were the tool that young Newton deployed in order to deal with these mechanical (transcendental) curves.

1.4 Fluxions

A Newtonian memorandum, written more than fifty years after the momentous intellectual revolution it describes, gives an account, basically confirmed by manuscript evidence, of his early mathematical discoveries:

> In the beginning of the year 1665 I found the Method of approximating series & the Rule for reducing any dignity of any Binomial into such a series. The same year in May I found the method of Tangents of Gregory & Slusius, & in November had the direct method of fluxions & the next year in January had the theory of Colours & in May following I had entrance into y[e] inverse method of fluxions. And the same year I began to think of gravity extending to y[e] orb of the Moon ... . All this was in the two plague years of 1665–1666. For in those days I was in the prime of my age for invention & minded Mathematicks & Philosophy more then at any time since.[14]

There would be much to say to decipher Newton's words and place them in context. For instance, the task of commenting on the meaning of the term philosophy would require space and learning not at my disposal.

commonly dubbed 'mechanical,' and with their assistance problems on these curves are solved no differently than in simpler curves." MP, 4, p. 559. "Quinetiam ad curvas lineas vulgo dictas Mechanicas inveniri possunt aequationes convergetes et earum beneficio problemata in his curvis non aliter solvi quam in curvis simplicioribus." MP, 4, p. 558.

[13] MP, 3, p. 79. "quantitates ... quae nullâ ratione geometricâ determinari et exprimi possunt, quales sunt areae vel longitudines curvarum." MP, 3, p. 78.

[14] Add. 3968.41, f. 85r. This passage is contained in a draft (August 1718) of a letter that Newton intended for Pierre Des Maizeaux. It is discussed in Westfall, *Never at Rest* (1980), p. 143.

Note three things about this memorandum. (i) The "method of approximating series" is the method of series expansion via long division and root extraction. Newton achieved also other methods for expanding y as a fractional power series in x, when the two variables are related by an algebraic equation. These methods, later generalized by Victor-Alexandre Puiseux, allowed Newton to go beyond the limitations of the common analysis, where finite equations were deployed, and express certain curves locally in terms of infinite fractional power series, which Newton called infinite equations.[15] (ii) The "rule for reducing any dignity of any binomial" is now called the binomial theorem for fractional powers, which Newton attained in winter 1664 by interpolating results contained in Wallis's *Arithmetica Infinitorum* (included in *Operum Mathematicorum Pars Altera* (1656)).[16] Such methods of series expansion were crucial for attaining two goals: the calculation of areas of curvilinear surfaces and the rectification of curves. (iii) Newton does not talk about discovering theorems, but rather methods and a rule. This last fact is of utmost importance because it reveals that, in his view, his results belonged to the analytical, heuristic stage of the method of problem solving.

In October 1666, Newton gathered his early results in a tract whose *incipit* reads "To resolve Problems by Motion these following Propositions are sufficient."[17] He conceived this tract as devoted to a method of resolution (i.e., "analysis") of geometrical problems, which makes use of the concept of geometrical magnitudes as generated by motion. This method, referred to in Newton's memorandum as the "direct and inverse method of fluxions," is discussed in part III. Note that the inverse method was always conceived by Newton as deeply intertwined with the method of approximating series and with the binomial rule.[18]

1.5 In the Wake of the *Anni Mirabiles*

In 1669 the first challenge arrived for the young mathematician. A slim book entitled *Logarithmotechnia*, printed in 1668, the work of the German Nicolaus Mercator, came to his attention. What Newton saw was worrying. Mercator had used an infinite equation (in our terms, a power series expansion of $y = 1/(1 + x)$) in order to square the hyperbola (i.e., calculate the area of the surface subtended by the hyperbola). This result belongs both to pure mathematics and to practi-

[15] These techniques are discussed in many treatises on algebraic curves: e.g., Brieskorn and Knörrer, *Plane Algebraic Curves* (1986), pp. 370ff.
[16] MP, 1, pp. 89–142.
[17] The "October 1666 Tract on Fluxions" is Add. 3958.3, ff. 48v–63v, and is edited in MP, 1, pp. 400–48.
[18] For a recent evaluation of Newton's early work on series and fluxions in the period 1664–1666, see Panza, *Newton et les Origines de l'Analyse: 1664–1666* (2005).

cal applications. It is, in fact, useful in facilitating the calculation of logarithms, a need deeply felt by seventeenth-century practitioners in all fields from navigation to astronomy. This was one of the results that Newton had achieved via binomial expansion or long division. He was able to do much more than this and therefore summarized his results regarding infinite series applied to quadrature in a small tract entitled *De Analysi per Aequationes Numero Terminorum Infinitas* (1669).[19]

Barrow, who was well informed about Newton's discoveries, immediately sent *De Analysi* to a mathematical practitioner called John Collins. The choice could not have been happier. Collins was at the center of a network of British and Continental mathematicians whom he kept up to date with an intense and competent correspondence. After taking copies of *De Analysi*, Collins informed a number of his correspondents about Newton's discoveries. He also made Newton aware of the Scotsman James Gregory (or Gregorie), who was pursuing researches on series expansions at a level comparable to what could be found in *De Analysi*. Collins's correspondence was the vehicle that allowed Newton to establish his reputation as a mathematician. Collins's network overlapped with that of the Royal Society; its president, William Brouncker, the secretary, Henry Oldenburg, and Wallis were certainly interested in Newton's mathematical researches on infinite series.

In 1672, Newton was elected a Fellow of the Royal Society because of the construction of the reflecting telescope, not because of his mathematics. And it was because of his ideas concerning the role of mathematics in natural philosophy that he initially found himself in a difficult relationship with the Royal Society. When he presented his 1672 paper on the nature of light, Newton made it clear that the undisputable certainty of his "new theory about light and colors" was guaranteed by mathematical reasoning. This thesis displeased the secretary, Henry Oldenburg, and the curator of experiments, Robert Hooke, who refrained from subscribing to what they perceived as a dogmatic position (see chapter 2). Newton found himself embroiled in a dispute that led him, after some years of tiresome correspondence with critics, to be reluctant about printing his philosophical ideas. Famously, in a different context, he was to complain about philosophy as an "impertinently litigious Lady."[20] What is relevant here is that, in the mid-1670s, much to Collins's frustration, he withdrew from any project of printing his mathematical discoveries on series and fluxions.

As I argue in Part VI, Newton's policy of publication is consistent with his self-portraiture as a natural philosopher who—contrary to the skeptical probabilism endorsed by many virtuosi of the Royal Society—could attain certainty thanks to

---

[19] See chapter 7 for further information.

[20] Newton to Halley (June 20, 1686) in *Correspondence*, 2, p. 437.

mathematics. Printing the algebraic, heuristic method would have exposed him to further criticisms; what he aimed at was certainty, and this was guaranteed by geometry. The algebraic analysis—as he later said to David Gregory—was "entirely unfit to consign to writing and commit to posterity."[21] To appreciate Newton's views on mathematics, one should not underestimate how sharp a boundary he drew in contrast with his mathematical practice between algebra and geometry, and how strongly he believed that only geometry could provide a certain and therefore publishable demonstration.

Newton's perception of the two layers of algebraic analysis and geometrical synthesis is already evident in his *Tractatus de Methodis Serierum et Fluxionum*, composed in 1670–1671.[22] The beginning of this long treatise is occupied by a revision and expansion of *De Analysi*. In the remaining twelve sections (labeled as problems) Newton "methodized" his researches into fluxions that he had first laid down in the October 1666 tract.[23] Here he developed the *analytical* method of fluxions, which was divided into two parts: (i) the direct method (mainly calculations of tangents and curvatures) and the inverse method (mainly calculations of areas and rectifications of curves). *De Methodis* ends with extensive tabulations of areas of surfaces subtended to curves. Newton soon developed (in an "Addendum" written in 1671) the idea that a *synthetic* form of the method of fluxions was required (see chapter 9). This more rigorous version, where no infinitesimals occur, was based on limit concepts and geometrical-kinematical conceptions and was systematized in a tract entitled "Geometria Curvilinea," written about 1680. The synthetic method of fluxions—the method of first and ultimate ratios—informs most of the *Principia* (1687).

Thus, in 1671—just after the completion of *De Methodis*, a summa of his analytical researches on series and fluxions—Newton began to rethink the status of his early researches, which are based on heuristic analogies and the use of infinitesimals, namely, on techniques that are far from the standards of exactness that he aimed at as a natural philosopher. In the 1670s he spent great effort in systematizing them, in rethinking their foundation, and in attempting alternative approaches. Several factors contributed to the more mature phase of Newton's mathematical production that followed the creative burst of the *anni mirabiles*. I note a few of these factors in the next section and elaborate on them in subsequent chapters.

---

[21] "Algebram nostram speciosam esse ad inveniendum aptam satis at literis posterisque consignandum prorsus ineptam." University Library Edinburgh MS Gregory C42, translated by D. T. Whiteside in MP, 7, p. 196. See also *Correspondence*, 3, p. 385.
[22] See chapter 8 for further information.
[23] "partly upon Dr Barrows instigation I began to new methodiz ye discourse of infinite series." Newton to Collins (July 20, 1670) in *Correspondence*, 1, p. 68.

1.6 Maturity

Young men should prove theorems, old men should write books.[24]

In 1669, Newton was elected Lucasian Professor, in succession to and thanks to the patronage of Barrow. He began preparing a first set of lectures on optics in which he claimed that certainty in natural philosophy can be guaranteed by the use of geometry (see chapter 2). A concern with certainty in mathematical method thus emerged in the context of Newton's early optical researches and remained anchored to them until maturity when, in the last Query 23/31 (1706/1717) of the *Opticks*, he wrote a famous peroration in favor of the use of the method of analysis and synthesis in natural philosophy. The investigation of difficult things, he claimed, could be pursued in natural philosophy only by following the steps of the mathematicians' method of enquiry. Newton wished to validate his natural philosophy mathematically, outstripping the skeptical probabilism that was rampant in his day, as he complained. Synthesis, not analysis, was the method that could guarantee the level of accuracy and certainty required for such an ambitious task. Further, as a successor of Barrow in the Lucasian Chair, Newton probably felt that his new status implied delivering mathematics in rigorous and systematic form. He began writing mathematical treatises characterized by length, maturity, and apparent uselessness (they seldom went to the press).[25]

Newton's involvement in preparing his next set of lectures on algebra led him to conceive the idea that analysis could also be approached differently from the way promoted by the moderns: in short, there could be a geometrical analysis, a geometrical rather than an algebraic method of discovery. Up to this point in this chapter, I have somewhat incorrectly equated analysis with algebra, and synthesis with geometry. But it is necessary to avoid such equivalences because they were not accepted by Newton and by many of his contemporaries. Not only synthesis but also analysis could be geometrical.

Lucasian Lectures on Algebra stemmed from a project on which Newton had embarked since the fall of 1669, thanks to the enthusiasm of John Collins: the revision of Mercator's Latin translation of Gerard Kinckhuysen's Dutch textbook on algebra. Newton's involvement in this enterprise was an occasion to rethink the status of common analysis. He began experimenting with what he understood as ancient analysis, a geometrical method of analysis or resolution that, in his opinion, the ancients had kept hidden. In his *Lucasian Lectures on Algebra*, which he deposited in the University Library of Cambridge in 1684 and from which William

[24] Godfrey H. Hardy, quoted by Freeman Dyson, in Albers, "Freeman Dyson: Mathematician, Physicist, and Writer" (1994), p. 2.
[25] But on Newton's attitude toward print publication versus manuscript circulation, see chapter 16.

Whiston edited the *Arithmetica Universalis* (1707), Newton extended Cartesian common analysis and arrived at new results in this field. But even in this eminently Cartesian text one can find traces of his fascination with the method of discovery of the ancients. The ancients, rather than using algebraic tools, were supposed to have a geometrical analysis that Newton wished to restore. This was a program shared by many in the seventeenth century. He also made it clear that synthesis, or composition, of geometrical problems had to be carried on—*contra* Descartes—in terms wholly independent of algebraic considerations (see chapter 4). The fascination with ancient analysis and synthesis, a better substitute, he strongly opined, for Cartesian common analysis (algebra) and synthesis (the techniques on the construction of equations prescribed by Descartes), prompted Newton to read the seventh book of Pappus's *Collectio* (composed in the fourth Century A.D. and printed alongside a Latin translation in 1588). He became convinced that the lost books of Euclid's *Porisms*, described incompletely in Pappus's synopsis, were the heart of the concealed ancient, analytical but entirely geometrical method of discovery (see chapter 5).[26]

Newton intertwined this myth of the ancient geometers with his growing anti-Cartesianism. In the 1670s he elaborated a profoundly anti-Cartesian position, motivated also by theological reasons. He began looking to the ancient past in search for a philosophy that would have been closer to divine revelation. The moderns, he was convinced, were defending a corrupt philosophy, especially those who were under Descartes' spell. Newton's opposition to Cartesian mathematics was strengthened by his dislike for Cartesian philosophy. Descartes in the *Géométrie* had proposed algebra as a tool that could supersede the means at the disposal of Euclid and Apollonius. Newton worked on Pappus's *Collectio* in order to prove that Descartes was wrong. He claimed that the geometrical analysis of the ancients was superior to the algebraic of the moderns in terms of elegance and simplicity. In this context, Newton developed many results in projective geometry and concerning the organic description of curves. His great success, achieved in a treatise entitled "Solutio Problematis Veterum de Loco Solido" (late 1670s) on the "restoration of the solid loci of the ancients," was the solution by purely geometrical means of the Pappus four-lines locus. This result, much more than the new analysis of infinite series and fluxions, pleased Newton because it was in line with his philosophical agenda (see chapter 5).

The importance of projective geometry emerged also in the study of cubics, when Newton found that these algebraic curves can be subdivided into five projective classes. His interest in the classification of cubic curves dates to the 1660s, but it was only in the late 1670s that, by deploying advanced algebraic tools, he achieved

---

[26] It seems that Newton did not know that Descartes expressed similar views in the "Responsio ad Secundas Obiectiones" in *Meditationes de Prima Philosophia* (1641) (AT, 7, pp. 155–6).

the array of results that later, in the mid-1690s, were systematized in *Enumeratio Linearum Tertii Ordinis*, a work that first appeared in print as an appendix to the *Opticks* (1704) (see chapter 6).

One should not forget another factor that determined Newton's option for geometry in the 1670s: the encounter with Huygens's *Horologium Oscillatorium*. In his masterpiece, printed in 1673, Huygens had employed proportion theory and *ad absurdum* limit arguments (method of exhaustion) and had spurned as far as possible the use of equations and infinitesimals (in his private papers he did employ symbolic infinitesimalist tools, but he avoided them in print).[27] Huygens offered an example to Newton of how modern cutting-edge mathematization of natural philosophy could be presented in a form consonant with ancient exemplars. The Lucasian Professor immediately acknowledged the importance of Huygens's work, and one might surmise that his methodological turn of the 1670s—which in part led him to cool his relationship with Collins and avoid print publication of his youthful algebraic researches—was related not only to a reaction against Cartesianism, but also to an attraction toward Huygens's mathematical style.

When Newton composed the *Principia*, in 1684–1686, he had a panoply of mathematical methods in his toolbox, methods that he could deploy in the study of force and motion. He gave pride of place to the synthetic method of fluxions (first elaborated in a treatise composed about 1680 and entitled "Geometria Curvilinea"), claiming in Section 1, Book 1, that this was the foundation on which the *magnum opus* was based. But in several instances, as a close reading of the text of the *Principia* makes clear, he appealed to quadrature techniques that belong to his algebraized new analysis. These quadratures were not, however, made explicit to the reader. Newton chose instead to insert in the body of the text a treatment of ancient analysis and its application to the solution of the so-called Pappus problem. In Part IV I discuss the policy of publication that led Newton to structure the text and the subtext of the *Principia* in ways consonant with his views on mathematical method.[28]

After the publication of the *Principia*, Newton ceased to be an isolated Cambridge professor. He had to defend and establish his rising position in the political and cultural world of the capital, where he moved in 1696 as Warden of the Mint. A first challenge, in 1691, from David Gregory (§8.5.1) on quadrature techniques induced him to work hard in the early 1690s on the composition of a treatise, *Tractatus de Quadratura Curvarum*, which appeared in 1704 as an appendix to the *Opticks*. *De Quadratura* opens with an introduction in which Newton claims that the method of fluxions is based on a conception of magnitudes generated by motion

---

[27] Yoder, *Unrolling Time* (1988).
[28] See also Guicciardini, *Reading the Principia* (1999).

and on limiting procedures that are consonant with the methods of the ancients (see chapter 9).[29]

Newton's growing fascination with the myth about the *prisca sapientia*, a pristine superior wisdom of the ancients, that characterizes his thought after the publication of the *Principia* resonates with his extensive researches on the ancient analysis that he carried on in the 1690s and early 1700s. His aim was to show that the youthful analytical method of fluxions could be reformulated in terms acceptable by ancient standards. He even explored a totally new method of discovery and proof. Newton left hundreds of manuscript pages, which culminated in an unfinished "Geometriae Libri Duo," devoted to his attempts to write a treatise on projective geometry written in a style reconstructed following the authority of Pappus (see part V). These aborted attempts are the more philosophy-laden texts belonging to Newton's mathematical *Nachlass*, since he made a deep effort to clarify the relations among the various sectors of his mathematical method: analysis, synthesis, algebra, geometry, mechanics, and natural philosophy. These terms have been used in this first chapter in an improperly ambiguous way. But commenting on Newton's works on method in subsequent chapters will allow me to clarify this terminology and decode Newton's somewhat arcane mode of expression.

Newton encouraged his acolytes to pursue researches in ancient analysis and never missed the opportunity for praising those, such as Huygens, who resisted the prevailing taste for the symbolism of the moderns, the "bunglers in mathematics."[30] When the polemic with Leibniz exploded, he could deploy his classicizing and anti-Cartesian theses against the German (see part VI). Thus, Newton's last mathematical productions, publications, and (often anonymous) polemical pieces were driven by a philosophical agenda difficult to reconcile with his mathematical practice.

[29] The other appendix, *Enumeratio Linearum Tertii Ordinis*, was also written in the 1690s, deploying notes on cubics dating from the 1670s. See chapter 6.

[30] Hiscock, *David Gregory, Isaac Newton and Their Circle* (1937), p. 42.

2 Newton on Certainty in *Optical Lectures*

> Newton then does stand somewhat apart not only in accomplishment but also in his philosophy of science.
>
> —Barbara Shapiro, 1983

The appointment to the Lucasian Chair (1669) led Newton to ponder over his role as a mathematician. Now he was not just a young creative protégé of Barrow but rather a professor who would soon address the Royal Society with new theories concerning the nature of light. That Newton began investigating the role of mathematics in natural philosophy is evident in his first set of lectures on optics, which he supposedly delivered between January 1670 and the end of Michaelmas term in 1672. As Lucasian Professor, Newton had to deliver one lecture each week during term and to deposit at least ten lectures every year. There thus exist a set of his lectures on optics and a set on algebra (§4.1).[1]

In his third lecture Newton stated a program that remained a leitmotif all his life:

> Thus although colors may belong to physics, the science of them must nevertheless be considered mathematical, insofar as they are treated by mathematical reasoning. Indeed, since an exact science of them seems to be one of the most difficult that philosophy is in need of, I hope to show—as it were, by example—how valuable is mathematics in natural philosophy. I therefore urge geometers to investigate nature more rigorously, and those devoted to natural science to learn geometry first. Hence the former shall not entirely spend their time in speculations of no value to human life, nor shall the latter, while working assiduously with an absurd

---

Epigraph from Barbara Shapiro, *Probability and Certainty in Seventeenth-Century England* (1983), p. 58.

[1] An early version of the *Principia* was also deposited by Newton. One should not assume that Newton actually delivered these lectures to his students (if he had any). Further, the dates on the deposited manuscripts were added in retrospect. The critical edition is in Newton, *The Optical Papers of Isaac Newton: Vol.1. The Optical Lectures 1670–1672* (1984), pp. 46–279, which includes both an earlier version of the lectures and the *Opticae*, pars 1 and 2, which Newton deposited in October 1674 in compliance with the statutes of the Lucasian Chair. The deposited lectures appeared posthumously, first in an English translation of pars 1 as *Optical Lectures: Read in the Publick Schools of the University of Cambridge* (1728) and then the complete Latin text as *Lectiones Opticae* (1729). Both these posthumous publications (based on a collation of a transcript and the original) have a complicated story, which is discussed in Alan Shapiro's commentary in Newton, *The Optical Papers* (1984), pp. 20–5. A facsimile of the first version of the lectures (MS Add. 4002 (Cambridge University Library)) can be found in Newton, *The Unpublished First Version of Isaac Newton's Cambridge Lectures on Optics* (1973).

> method, perpetually fail to reach their goal. But truly with the help of philosophical geometers and geometrical philosophers, instead of the conjectures and probabilities that are being blazoned about everywhere, we shall finally achieve a science of nature supported by the highest evidence.[2]

Ideas very similar to these were to occur in the *Preface* to the second edition of the *Principia* (1713), written by Roger Cotes some forty years later. Alan Shapiro has emphasized the pervasive role of the quest for certainty in Newton's thought and has demonstrated that Newton, very early in his intellectual career, conceded that natural philosophy, even when approached mathematically, cannot reach the absolute certainty of geometry. But Newton made it clear that geometers who practice natural philosophy are able to go a long way beyond the modest aim of the experimental naturalists, namely, conjectures and probabilities. Shapiro wrote that in the last years of his life Newton changed his mind about the possibility of philosophical geometers and geometrical philosophers being able to approach absolute certainty:

> Although he [Newton] soon modified his excessive claims and more carefully distinguished mathematical demonstrations from less than certain experimental conclusions, he continued to prize mathematical theories and seek "truth" and "certainty," words that appear frequently in his writings. Only in the last decades of his life did he accept the probabilism of his contemporaries.[3]

---

[2] Newton, *The Optical Papers* (1984), pp. 87, 89, and 437, 439. "Sic etiamsi colores ad Physicam pertineant, eorum tamen scientia pro Mathematica habenda est, quatenus ratione mathematica tractantur. Imo vero cum horum accurata scientia videatur ex difficillimis esse quae Philosophus desideret; spero me quasi exemplo monstraturum quantum Mathesis in Philosophia naturali valeat; et exinde ut homines Geometras ad examen Naturae strictius aggrediendum & avidos scientiae naturalis ad Geometriam prius addiscendam horter: ut ne priores suum omnino tempus in speculationibus humanae vitae nequaquam profuturis absumant, neque posteriores operam praepostera methodo usque navantes, a spe sua perpetuo decidant: Verum ut Geometris philosophantibus & Philosophis exercentibus Geometriam, pro conjecturis et probabilibus quae venditantur ubique, scientiam Naturae summis tandem evidentijs firmatam nanciscamur." Newton, *The Optical Papers* (1984), pp. 86, 88, and 436, 438. It should be noted that Galileo in the *Dialogo*, which Newton knew in Thomas Salusbury's translation (see Newton, *Certain Philosophical Questions: Newton's Trinity Notebook* (1983), p. 202, and Salusbury, *Mathematical Collections and Translations in Two Tomes* (1661–65)), referred to a "filosofo geometra" who is able to apply mathematics to the study of nature: "così, quando il filosofo geometra vuol riconoscere in concreto gli effetti dimostrati in astratto, bisogna che difalchi gli impedimenti della materia; che se ciò saprà fare, io vi assicuro che le cose si riscontreranno non meno aggiustamente che i computi aritmetici. Gli errori dunque non consistono né nell'astratto né nel concreto, né nella geometria né nella fisica, ma nel calcolatore, che non sa fare i conti giusti." Galilei, *Dialogo Sopra i Due Massimi Sistemi del Mondo* (1998), 2, pp. 511.3–5. Similarly in *Auctoris Praefatio* to the *Principia*, Newton stated that errors are not due to the imperfections of geometry and mechanics but rather to the artificer who applies them. See chapter 13.

[3] Alan Shapiro, *Fits, Passions, and Paroxysms* (1993), p. 14. See also Alan Shapiro, "Newton's 'Experimental Philosophy'" (2004).

It is difficult to assess the reasons and influences behind the rather extremist methodological position that Newton endorsed in his youth. Barrow might have instilled in the young Newton the conviction that mathematics not only possesses certainty but can also transfer it to the fields to which it is applied. But Newton's position is passionate and idiosyncratic and goes beyond anything that Barrow might have envisaged. Newton's claims concerning mathematical certainty were perceived as alien by most of his contemporaries. In order to defend his somewhat isolated position Newton had to clarify two issues: (i) why mathematics could be considered a source of certainty, and (ii) how mathematics could transfer its certainty to natural philosophy.

In this chapter I do not deal with Newton's approach to these issues. It is hard to find a justification for his program in the writings of the early 1670s. Actually, Newton's critics often complained about his lack of justification and explanation, and the dogmatism with which he defended his method. But Newton soon faced these issues in the context of his researches *contra* Cartesian mathematics (see part II). It is only at the end of this book that one will be in a position to appreciate the negotiations and strategies that Newton had to engineer during his long intellectual career to assess the certainty of mathematics and mathematized natural philosophy vis à vis his mathematical practices, which were based on innovative, bold, heuristic, and far from well-settled methods.

In this chapter I describe whom the young Lucasian Professor was criticizing, what his models were, and the aims he set himself when proposing to inject geometry into natural philosophy in his search for "highest evidence." My aim is to locate Newton's early program in a dialogical context. His idiosyncratic philomathematism is, I believe, a critical response, still lacking elaboration and justification, against a position that was "being blazoned about" in the influential circle of the natural philosophers gathered at the Royal Society.

One of Newton's polemical targets might be identified as Descartes; indeed, anti-Cartesian feelings pervade Newton's work. In his natural philosophy Descartes had adopted what is often called a hypothetical physics whereby hypothetical mechanical models are introduced a priori in order to deduce the observed phenomena. Descartes' explanation of the magnet has become an icon of this sort of hypothetical mechanicism (figure 2.1). Descartes conceived of the magnet as emitting screw-shaped particles that penetrate the pores of matter and thus exert a circulatory motion ultimately accounting for the attraction between magnet and iron in terms of contact action between corpuscles. This model was proposed as a hypothesis based on local mechanical interactions that, although plausible, might not occur in nature. Indeed, in Cartesian physics more than one model could be invoked in order to reach a mechanical explanation of the same phenomenon. According to Newton, Descartes' hypothetical models were not derived from observation; they were assumed from first principles. Newton disliked these characteristics of Cartesian hypotheticism because, in his opinion, Descartes' models were not derived

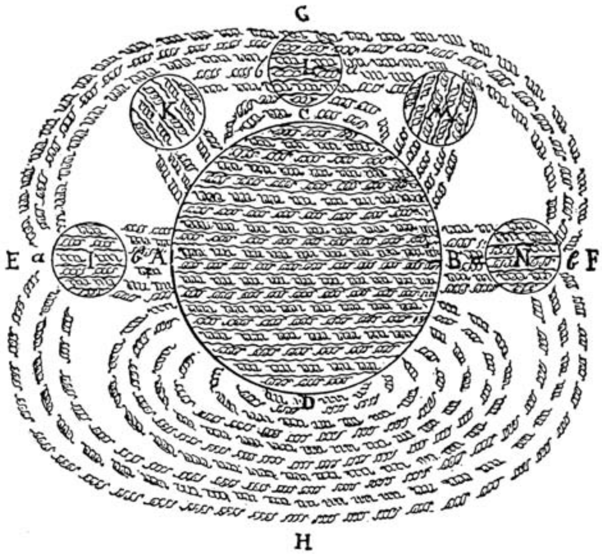


Figure 2.1
The earth as a giant magnet, according to Descartes. The screw-shaped particles are emitted from the North Pole and travel toward the South Pole, generating magnetic effects by direct contact with other bodies. Source: Descartes, *Principia Philosophiae* (1644), p. 271. Courtesy of the Biblioteca Angelo Mai (Bergamo).

from experiments ("deduced from the phenomena")[4] but proposed as instances of one of many plausible causal explanations of phenomena.

Newton's critical reading of Descartes can appear reductive to Cartesian scholars. Indeed, Descartes had much to say about the role of mathematics, especially geometry, in natural philosophy. Geometry was the model of certain knowledge for Descartes. Further, reference to geometry was crucial in grounding Descartes' conception of matter as extension, his rejection of atomism, his view of the phenomena of nature as understandable in terms of matter and motion. It is by starting from a geometrized natural philosophy that Descartes drew such crucial consequences as the nonexistence of a vacuum. As in the case of Descartes the mathematician, Newton approached Descartes the natural philosopher from a very restrictive and biased point of view. What Newton kept in critical view was the genre of hypothetical-deductive modeling that informs so much of Descartes' *Principia Philosophiae* (1644), a book that was to be read, according to its author, as

[4] See, e.g., the famous passage in the Scholium generale in Newton, *Principles*, p. 943. See also §14.3.2.

a "romance."[5] Indeed, Descartes made clear that the corpuscular models that he proposed as explanations of a wide variety of phenomena were to be received as plausible mechanical accounts of their functioning and genesis. Each one of these explanations, taken separately, was neither unique nor certain, but their cumulative effect, according to Descartes, delivered a powerful defense of the viability of a mechanistic view of nature.[6]

But when expressing his dissatisfaction towards "conjectures and probabilities" Newton might well have had in mind his contemporaries: the virtuosi of the Royal Society associated with the experimental researches of Robert Hooke and Robert Boyle. As Barbara Shapiro has argued, most of the English intellectuals associated with the Royal Society "were deeply concerned with matters empirical, and they concluded that neither syllogism nor mathematical reasoning was an appropriate vehicle for ordering the data they were collecting."[7] The certainty of Aristotelian logic and Euclidean geometry was deemed by them to be extraneous to experimental philosophy, where the best one could reach was probability through a patient collection of facts. The language that Hooke and Boyle promoted was one of avoiding both dogmatic certainty and extreme skepticism. Their position can be characterized as a form of mitigated skepticism according to which it is accepted that in natural philosophy one can attain probability but not absolute certainty.[8] It is also true that Boyle praised mathematics and recognized that mathematical demonstration can yield certainty.[9] He valued quantitative reports of experimental results when they could be attained. Mathematics occupies a prominent place in Boyle's thought, but he expressly delimited its sphere of influence. Typically, in *Hydrostatical Paradoxes* (1666) and *Medicina Hydrostatica* (1690), he defended the idea that the "exactness" and "preciseness" of mathematics cannot be attained in "experiments where we are dealing with gross matter."[10]

[5] As Descartes famously stated in a letter to the abbé Claude Picot (translator of the first French edition of *Principia Philosophiae*) that Newton might have read in Descartes, *Opera Philosophica* (1656), which he possessed. See Harrison, *The Library of Isaac Newton* (1978), no. 506.

[6] The literature concerning Descartes is as vast as the one concerning Newton. Two scholarly introductions are Garber, *Descartes' Metaphysical Physics* (1992) and Gaukroger, *Descartes' System of Natural Philosophy* (2002). For a recent examination of Descartes' experimental method, see Buchwald "Descartes's Experimental Journey" (2008).

[7] Barbara Shapiro, *Probability and Certainty in Seventeenth-Century England* (1983), p. 5.

[8] On constructive or mitigated skepticism, see Popkin, *The History of Scepticism* (2003), pp. 112ff.

[9] Boyle, *Of the Usefulness of Mathematicks to Natural Philosophy* (1671). This was part of *Some Considerations Touching the Usefulnesse of Experimental Natural Philosophy* (1671) and was reprinted in Boyle, *The Works of the Honourable Robert Boyle in Five Volumes* (1744), 3, pp. 392–456.

[10] "[Some readers] will not like that I should offer for proofs such physical experiments, as do not always demonstrate the things, they would evince, with a mathematical certainty and accurateness. . . . In physical enquiries it is often sufficient, that our determinations come very near the matter,

Newton positioned himself against the Baconian inductivist program in vogue at the Royal Society as much as against the Cartesian rationalist hypotheticism. From his many pronouncements on method one can infer that he considered both Bacon's bottom-up experimental procedure and Descartes' top-down method doomed to yield only probability. Bacon's proposed patient collection of numerous experiments was not sufficient to guarantee certainty. Descartes alignment of clear and distinct ideas could only deliver, as Newton wrote in his maturity, "little better then a romance."[11]

Newton proposed to overcome the probabilistic outcomes of both Cartesianism and Baconianism by "proceeding alternately from experiments to conclusions & from conclusions to experiments"[12] (see chapter 14). As far as possible, general principles of philosophy had to be *deduced*, not *induced*, from experiments. A single well-chosen experiment (the *experimentum crucis*) or phenomenon (the planetary laws) could allow the geometrical philosopher to deduce the truth of a principle, or better, to initiate a process of approximation to the truth grounded on successive deductions of conclusions from experiments and experiments from conclusions.

Much of the polemic between Hooke and Newton over the 1672 paper on light (in which Newton proposed his two prisms *experimentum crucis*) can be viewed as a clash between two different conceptions of method.[13] On the one hand, Newton claimed that through a single experiment he had been able to mathematically deduce a property of light beyond doubt (white light is composite; its components exhibit different refraction coefficients). On the other hand, Hooke denied Newton's claims to necessity and certainty, and advocated a methodology based on the evaluation of a plurality of experimental tests. In a passage that Henry Oldenburg, the secretary of the Royal Society, cautiously censored, Newton went so far as to state,

> A naturalist would scearce expect to see ye science of those [colors] become mathematicall, & yet I dare affirm that there is as much certainty in it as in any other part of Opticks. For what I shall tell concerning them is not an Hypothesis but most rigid consequence, not conjectured by barely inferring 'tis thus because not

---

though they fall short of a mathematical exactness." Boyle, *Works* (1744), 2, p. 741. "[I do not] pretend (and indeed it is not necessary) that the proportion, obtainable by our method, should have a mathematical preciseness. For in experiments where we are to deal with gross matter, and to employ about it mathematical instruments, it is sufficient to have a physical, and almost impossible to obtain (unless sometimes by accident) a mathematical exactness." Boyle, *Works* (1744), 5, p. 480. Also available in *The Works of Robert Boyle* (1999–2000).

[11] "But if without deriving the properties of things from Phaenomena you feign Hypotheses & think by them to explain all nature you may make a plausible systeme of Philosophy for getting your self a name, but your systeme will be little better then a Romance." Add. 3970, f. 480v. Discussed in Alan Shapiro, "Newton's 'Experimental Philosophy'" (2004), pp. 195–6.

[12] Ibid.

[13] Bechler, "Newton's 1672 Optical Controversies" (1974).

> otherwise or because it satisfies all phaenomena (the Philosophers universall Topick,) but evinced by ye mediation of experiments concluding directly & wthout any suspicion of doubt.[14]

Hooke, who could read this passage canceled from the paper printed in the *Philosophical Transactions*, remarked,

> Nor would I be understood to have said all this against his [Newton's] theory as it is an hypothesis, for I doe most Readily agree with him in every part thereof, and esteem it very subtill and ingenious, and capable of salving all the phaenomena of colours; but I cannot think it to be the only hypothesis; not soe certain as mathematicall Demonstrations.[15]

Hooke was hitting upon themes that were part of the ideology of the Royal Society, defended by influential members such as Boyle, Joseph Glanvill, and Thomas Sprat. Newton's quest for mathematical certainty could appear to them as mere arrogance and as a departure from the Baconian program that informed their agenda. Hooke explicitly invoked Bacon's authority against Newton in his letter of June 1672 to Lord Brouncker, the president of the Royal Society:

> I see noe reason why Mr. N. should make soe confident a conclusion that he to whome he writ did see how much it was besides the business in hand to Dispute about hypotheses. For I judge there is noething conduces soe much to the advancement of Philosophy as the examining of hypotheses by experiments & the inquiry into Experiments by hypotheses, and I have the Authority of the Incomparable Verulam to warrant me.[16]

There was a moral dimension to the Royal Society's initial rejection of Newton's self-enrollment in the camp of geometrical philosophers: such a program was against the precept that "true knowledge is modest and *wary*; 'tis ignorance that is so bold and presuming." From this viewpoint the certainty of geometry should not be traded for the experimental practice that was promoted as a cooperative effort, where different explanations could and should be confronted without any dogmatic emphasis. The probabilism and moderate skepticism that were widespread in Newton's England were part of a much larger movement toward tolerance endorsed by the Fellows of the Royal Society. Newton's geometrical philosopher could seem too much akin to what Glanvill termed a "dogmatist [who] betrays a poverty and a narrowness of spirit [and, being] too confident in opinions, [shows] ill manners and immodesty."[17]

[14] *Correspondence*, 1, pp. 96–7.
[15] *Correspondence*, 1, p. 113.
[16] *Correspondence*, 1, p. 202.
[17] Glanvill, *Scepsis Scientifica* (1665), p. 195.

Feingold showed that Newton's approach, consisting in an "uncompromising conviction concerning the primacy of mathematics in the domain of natural philosophy ... and his condescending view of the natural history tradition," was shared by only a fraction of the Royal Society's Fellows for many years to come.[18] During Newton's presidency, many of the Fellows defined themselves as naturalists and opposed the Newtonian philomaths. Feingold brought new insight into the dispute that was fought between the two camps, and he documented the tension that existed among the Fellows of the Royal Society. The quarrel over Newton's 1672 paper on the *experimentum crucis* can be seen as the beginning of this struggle, which was still raging the year of Newton's death (1726), in the battle between Martin Folkes and Hans Sloane for the presidency of the Royal Society. Feingold opposed the idea that the distinction between the two groups could be defined in terms of scientists versus amateurs. He warned that "by the early eighteenth century the cleavage between a group comprised primarily of mathematicians, astronomers, and physicists, on the one hand, and naturalists, physicians, and general scholars, on the other, was indicative of taste, not competence."[19] A towering figure in English natural philosophy such as Boyle showed a marked disinclination to merge mechanicism and mathematization, even when he recognized that his corpuscular view of matter allowed the description of the natural processes in geometrical terms. Both Boas Hall and Shapin showed that Boyle did so not because of mathematical illiteracy but because the language he promoted had to avoid unwarranted expectations of certainty and accuracy, and had to be accessible to the scrutiny of the largest number of inquirers. Modesty in theoretical assessment and accessibility were values deeply embedded in Boyle's moral depiction of the natural philosopher.[20]

It would be, however, narrow-minded to portray Newton as an outsider fighting single-handedly in favor of mathematized natural philosophy against a compact group of Baconian naturalists. It is true that Boyle sanitized experimental work from interference by mathematics, but alternative views emerged as well, for instance, in the works of Hooke, Christopher Wren, John Wallis, Isaac Barrow, and later Edmond Halley.

In most of his works Hooke displayed due reverence toward Boyle's mitigated skepticism. For instance, in the Preface to the *Micrographia* (1665) he made clear that his hypotheses, while "grounded and confirm'd" by experiment, should be taken "only as Conjectures and Quaeries" whose epistemological status was far from the mathematical certainty attained through "any Infallible Deductions or certainty

[18] Feingold, "Mathematicians and Naturalists" (2001), p. 78.
[19] Ibid., p. 94.
[20] Boas, *Robert Boyle and Seventeenth-Century Chemistry* (1958); Shapin, "Robert Boyle and Mathematics" (1988).

of Axioms." It is evident from the Preface of the *Micrographia* that Hooke paid attention to deeply felt values shared by his patrons:

> If therefore the reader expects from me any infallible deductions, or certainty of axioms, I am to say for myself, that those stronger works of wit and imagination are above my weak abilities; or if they had not been so, I would not have made use of them in this present subject before me: Wherever he finds that I have ventur'd at any small conjectures, as the causes of things that I have observed, I beseech him to look upon them only as doubtful problems, and uncertain guesses, and not as unquestionable conclusions, or matters of unconfutable science.[21]

The message delivered by Hooke, in a work that has been aptly described as a Royal Society venture,[22] could not be more at odds with Newton's defense of the role of geometrical philosophers. It should also be noted that in his *magnum opus* Hooke lost no occasion to praise Bacon for his empiricism and Descartes for his corpuscular hypothetical models.

However, Hooke was also interested in mathematical modeling, especially when it came to planetary motions. Recent research illuminates the extent of Hooke's mathematization in this field.[23] His employment of mathematics, however, was quite different from Newton's, both in content and role. Hooke's mathematization of the planetary orbits was preeminently graphic. It can be understood as a graphic simulation of the mechanical devices (the compound pendulum and a ball rolling on an inverted cone) that he devised in order to test his theories on central force motion. In Hooke's graphic mathematics is instantiated a characteristic of his work highlighted by both Bennett and Bertoloni Meli: an entanglement between art and nature, so that mechanical devices work as cognitive tools.[24] Hooke's mathematization was extraneous to Newton's much more abstract methods. However, some of the mathematical techniques that Newton deployed, especially those related to the organic description of curves, implied the use of mechanical devices that Newton described graphically and that he might have even constructed (§5.4.3).

The young Lucasian Professor found more consonance with the views held by Wren, Wallis, and especially Barrow, who in *Lectiones Mathematicae* (delivered in 1664–1666) rejected the distinction between sensible and intelligible, between pure and mixed (or concrete) mathematics, by stating that since continuous magnitude is the "affection" of all things, there is no part of "physics" that is not reducible to

[21] Hooke, *Micrographia* (1665), Preface [fifth page].
[22] Hunter, "Hooke the Natural Philosopher" (2003), p. 124.
[23] Nauenberg, "Hooke, Orbital Motion, and Newton's Principia" (1994); "Robert Hooke's Seminal Contributions to Orbital Dynamics" (2005).
[24] Bennett, "The Mechanics' Philosophy and the Mechanical Philosophy" (1986); Bertoloni Meli, *Thinking with Objects* (2006).

geometry. Barrow went so far as to claim that mathematics is "co-extended with physics"[25] (see chapter 8).

Newton's quest for certainty in a geometrized natural philosophy, while in the minority and oriented toward "immoral" dogmatism, found resonances in the works of a group of mathematicians interested in promoting mixed mathematics. Had Newton heard one of Wren's lectures at Gresham College, he could not but have agreed with the following:

> Mathematical Demonstrations being built upon the impregnable Foundations of Geometry and Arithmetick, are the only Truths, that can sink into the Mind of Man, void of all Uncertainty; and all other Discourses participate more or less of Truth, according as their Subjects are more or less capable of Mathematical Demonstration. Therefore, this rather than Logick is the great *Organ Organωn* of all infallible Science.[26]

Wren was aiming at "a real Science of Nature, not an Hypothesis of what Nature might be" by "a geometrical Way of reasoning from ocular Experiment."[27] Similarly, Barrow stated:

> Mathematicians ... only meddle with such things as are certain, passing by those that are doubtful and unknown. They profess not to know all Things, neither do they affect to speak of all Things. What they know to be true, and can make good by invincible Arguments, that they publish and insert among their Theorems. Of other Things they are silent and pass no Judgement at all, chusing rather to acknowledge their Ignorance, than affirm any Thing rashly. They affirm nothing among their arguments or Assertions which is not most manifestly known and examined with utmost Rigour, rejecting all probable Conjectures and little Witticism.[28]

Thus, in his *Optical Lectures* and in writings surrounding the 1672 paper on the new theory of light and colors, Newton aggressively took sides in favor of mathematicians like Wren and Barrow and against naturalists like Boyle. Why he endorsed this stance so passionately is difficult to fathom. Newton's main biographer, Richard Westfall, has done much to clarify why a "sober, silent, thinking lad" from Lincolnshire was consumed with searching for stability, truth, and certainty.[29] To Westfall's psychological analysis I would like to add an observation. The young

---

[25] "For magnitude is the common affection of all physical things, it is interwoven in the Nature of Bodies, blended with all corporeal Accidents"; "I say there is no part of this [Physics] which does not imply Quantity ... and consequently which is not in some way dependant on Geometry"; "Mathematics ... is adequate and co-extended with physics." Barrow, *The Usefulness of Mathematical Learning Explained and Demonstrated* (1734), pp. 21, 22, 26.
[26] Wren, *Parentalia* (1750), pp. 200–1.
[27] Ibid. On Wren, see Bennett, *The Mathematical Science of Christopher Wren* (1982), esp. pp. 118–20.
[28] Barrow, *The Usefulness of Mathematical Learning* (1734), p. 64.
[29] Westfall, *Never at Rest* (1980), pp. 40–65.

Newton perceived himself preeminently as a mathematician. The advances that he had achieved in the *anni mirabiles* told him that mathematics was his métier. The equally important advances in the theory of colors, he was convinced, had been possible only thanks to the application of mathematical method to experimental enquiry. He elaborated at length on this last point until the end of his life (see part V).

But if certainty was to be sought in mathematics, what was its method and what role did it have in natural philosophy? Or rather, which among the available alternative mathematical methods were exact and legitimate, and how could they be injected successfully into experimental discourse? Neither in *Optical Lectures* nor in the polemical writings addressed against the critics of the new theory of light did Newton elaborate an answer. In trying his way out of these challenging questions he found himself in a complex position. Most notably, the new methods that were developed in Newton's times, and to whose progress he contributed massively, departed from the standards of rigor set forth both by Greek tradition and by the new canon advanced by Descartes.

Because it is from Cartesian common analysis that Newton took inspiration as a young mathematician, and because it is against this tradition that he turned in later years, it is appropriate to provide in the next chapter a brief characterization of what Descartes had to say about exactness, demarcation between legitimate and illegitimate tools of construction, and simplicity, in an essay pamphlet printed as one of the appendices to *Discours de la Méthode* (1637) that fell into Newton's hands in 1664: the *Géométrie*.

3 Descartes on Method and Certainty in the *Géométrie*

> If, then, we wish to resolve any problem, we first suppose the solution already effected, and give names to all the lines that seem needful for its construction, to those that are unknown as well as to those that are known. Then, making no distinction between known and unknown lines, we must unravel the difficulty in any way that shows most naturally the relations between these lines, until we find it possible to express a single quantity in two ways. This will constitute an equation. ... This is one thing which I believe the ancient mathematicians did not observe, for otherwise they would not have put so much labor into writing so many books in which the very sequence of the propositions shows that they did not have a sure method of finding all, but rather gathered together those propositions on which they had happened by accident.
>
> —René Descartes, *Géométrie*, 1637

> What the ancients have taught is so scanty and for the most part so lacking in credibility that I may not hope for any kind of approach toward truth except by rejecting all the paths which they have followed.
>
> —René Descartes, *Les Passions de l'Âme*, 1649

3.1 Analysis and Synthesis in Pappus

Few works in the history of mathematics have been more influential than Descartes' *Géométrie* (1637). The canon defined in this revolutionary essay was to dominate the scene for many generations, and its influence on the young Newton cannot be overestimated. Indeed, most of Newton's mathematical work can be understood as a development of and a response to the *Géométrie*, which he came to know in its Latin translation (1659–1661) due to Frans van Schooten. It is thus essential to recall some salient aspects of this pivotal text.[1]

Epigraph sources: (1) Descartes, *Géométrie*, pp. 300 [6–9], 304 [17]. The first page number indicates the French text of *La Géométrie*, which appeared in Descartes, *Discours de la Méthode* (1637), pp. 297–413. The bracketed numbers indicate the pages in the English translation provided in the Dover facsimile edition. (2) "ce que les anciens en ont enseigné est si peu de chose, et pour la plupart si peu croyable, que je ne puis avoir aucune espérance d'approcher de la vérité qu'en m'éloignant des chemins qu'ils ont suivis." Descartes, *Les Passions de l'Âme* (1649), p. 207.

[1] Bos, *Redefining Geometrical Exactness* (2001) is an authoritative guide to the *Géométrie*. For different approaches, see also Molland, "Shifting the Foundations" (1976); Giusti, "La *Géométrie*

The difficulties that one encounters in historically evaluating the *Géométrie* are, in fact, momentous, and this preliminary chapter does not attempt to overcome them. First, the historian should place Descartes' mathematics in the context of his philosophical ideas. Newton's initial response to Cartesian epistemology and ontology has been documented most notably by James E. McGuire and Martin Tamny.[2] It does not seem to me, however, that Newton's reading of the *Géométrie* interacted with those philosophical concerns, which were certainly vital for Descartes. Therefore, I disregard this complex and deep issue. Second, the relation between tradition and innovation in Descartes' mathematical work is hard to define. From one point of view, the *Géométrie* marks a turning point in the elaboration of a theory of algebraic equations applied to the study of curves. From another point of view, its structure and language can be understood only by taking into consideration agendas that polarized the attention of early-seventeenth-century mathematicians. Newton was quick to exploit the innovative abstract algebraic theory elaborated by Descartes. But he manifested also a passionate interest in confronting himself with, and ultimately criticizing, the canon of problem solving proposed in the *Géométrie*, the canon that constitutes Descartes' response to debates on the nature of analysis and synthesis that were ignited by the publication of Pappus's *Collectio* in 1588. Paradoxically, when Newton devoted attention to this obsolescent facet of the *Géométrie*, he proved to be more conservative than Descartes.

Contrary to what one might believe, the *Géométrie* displays few elements of what is nowadays identified as analytic geometry, a theory that certainly would be unthinkable without Descartes' contribution, but that is still only *in nuce* in Cartesian geometry as Newton met in the *Géométrie* as a young student at Trinity College.[3] Most notably, in the *Géométrie* one can find neither Cartesian coordinate axes nor new curves plotted from their equations. Rather (oblique) coordinates are introduced by choosing certain directions embedded in the given figure, and curves are most often posited by geometrical definitions (e.g., when an ellipse is defined as a section of a plane with a cone) or introduced as traced by motion.[4] Equations are subsequently derived from the given figure and serve as symbolic devices useful for

---

di Descartes tra Numeri e Grandezze" (1990); Israel, "Dalle Regulae alla *Géométrie*" (1990); and Mancosu, *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century* (1996), pp. 65–91. Among the numerous French scholars who, continuing the tradition of Pierre Boutroux, Léon Brunschvigc, Pierre Costabel, Jules Vuillemin, Gilles-Gaston Granger, have contributed important works on the *Géométrie* are Serfati, "Les Compas Cartésiens" (1993), Rashed, "La *Géométrie* de Descartes et la Distinction Entre Courbes Géométriques et Courbes Mécaniques" (1997), Jullien, *Descartes: La Géométrie de 1637*.

[2] Newton, *Certain Philosophical Questions* (1983), pp. 127–94.

[3] Newton worked on the second Latin edition, but he might have encountered also the first Latin edition. In this book I cite from the second Latin edition, *Geometria* (Amsterdam: apud Ludovicum & Danielem Elzevirios, 1659–1661).

[4] For Descartes' conceptions concerning curves, their construction, and classification, see §3.2.3.

abbreviating geometrical relations. It should be stressed that the relation between equations and curves in Descartes has been much discussed in the literature, and is the object of much disagreement among Cartesian scholars.

It has been suggested by scholars such as Bos that in order to understand the language and the structure of the *Géométrie*, one must place it in the context of the problem-solving techniques elaborated by the early-seventeenth-century representatives of what Mahoney termed the analytical school: a group of mathematicians who united their competence in algebra with an innovative reading of the heuristic techniques of the ancient geometers.[5] Most notably, they referred to the Alexandrian mathematician Pappus (fl. A.D. 320) whose *Collectio Mathematica* appeared in Pesaro in 1588 thanks to the edition of the Greek text and Latin translation by Federico Commandino. The seventh book of the *Collectio* consisted in an incomplete presentation of works (mostly lost and no longer available to the readers of Commandino's translation) which, according to Pappus, had to do with a method followed by the ancient geometers: the method of analysis. In treating these works, Pappus assumed that his readers had access to them, his aim being that of introducing, commenting, and filling the gaps. For early-modern mathematicians it was an arduous and challenging task to divine the lost ancient works on analysis. The opening of the seventh book is often quoted. It is a passage, obscure to early-modern readers, whose decoding was at the top of the agenda of those numerous enthusiasts who were convinced that here lay hidden the key to the method of discovery of the ancients. Given the importance this passage had for Descartes, and for Newton, it is worth quoting at length:

> That which is called the *Domain of Analysis*, my son Hermodorus, is, taken as a whole, a special resource that was prepared, after the composition of the *Common Elements*, for those who want to acquire a power in geometry that is capable of solving problems set to them; and it is useful for this alone. It was written by three men: Euclid the Elementarist, Apollonius of Perge, and Aristaeus the Elder, and its approach is by analysis and synthesis.
>
> Now analysis is the path from what one is seeking, as if it were established, by way of its consequences, to something that is established by synthesis. That is to say, in analysis we assume what is sought as if it has been achieved, and look for the thing from which it follows, and again what comes before that, until by regressing in this way we come upon some one of the things that are already known, or that occupy the rank of a first principle. We call this kind of method "analysis," as if to say *anapalin lysis* (reduction backward). In synthesis, by reversal, we assume what was obtained last in the analysis to have been achieved already, and, setting now in natural order, as precedents, what before were following, and fitting them to each other, we attain the end of the construction of what was sought. This is what we call "synthesis."

[5] Bos, *Redefining Geometrical Exactness* (2001); Mahoney, *The Mathematical Career of Pierre de Fermat (1601–1665)* (1973), pp. 1–14.

> There are two kinds of analysis: one of them seeks after truth, and is called "theorematic": while the other tries to find what was demanded, and is called "problematic." In the case of the theorematic kind, we assume what is sought as a fact and true, then, advancing through its consequences, as if they are true facts according to the hypothesis, to something established, if this thing that has been established is a truth, then that which was sought will also be true, and its proof the reverse of the analysis; but if we should meet with something established to be false, then the thing that was sought too will be false. In the case of the problematic kind, we assume the proposition as something we know, then, proceeding through its consequences, as if true, to something established, if the established thing is possible and obtainable, which is what mathematicians call "given," the required thing will also be possible, and again the proof will be the reverse of the analysis; but should we meet with something established to be impossible, then the problem too will be impossible. Diorism is the preliminary distinction of when, how, and in how many ways the problem will be possible. So much, then, concerning analysis and synthesis.[6]

Pappus here makes a distinction between analysis and synthesis. Analysis (*resolutio*) was often conceived of as a method of discovery or problem solving that, working step by step backward from what is sought as if it had already been achieved, eventually arrives at what is known. Synthesis (*compositio*) goes the other way round: it starts from what is known and, working through the consequences, arrives at what is sought. On the basis of Pappus's authority it was often stated that synthesis reverses the steps of analysis. It was synthesis that provided the rigorous proof. Thus the widespread belief that the ancients had kept the method of analysis hidden and had published only the rigorous synthesis, either because they considered the former not wholly demonstrative, or because they wanted to hide the method of discovery. Such ideas concerning the ancients were shared by many, including François Viète and Descartes. The historian, of course, has to be careful in distinguishing approaches to the ancient tradition, which were considerably different. Viète, Descartes, and Newton mused on the classical tradition with different agendas in mind.

One should note that the Greek terms *analysis* and *synthesis* were interchangeable with the Latin *resolutio* and *compositio*, which were rendered in English as *resolution* and *composition*, and sometimes, especially in geometrical practice, with *resolution* and *construction*. These mathematical terms interacted in a complex way with the technical vocabulary pertaining to the philosophical, logical, chemical, and medical traditions.[7]

Pappus's description of the methods of analysis and synthesis has many ambiguities. In explaining what consequence means in the second paragraph of the foregoing

[6] Pappus, *Book 7 of the Collection* (1986), pp. 82–4.

[7] These interactions are particularly important when studying the logical tradition, exemplified by the work of Jacopo Zabarella (1533–1589).

quotation, Pappus tells that in searching for the consequences of something that is assumed, one should look for "the thing from which it follows." Apparently, consequence means a neighboring term in a deductive sequence, something that follows from a term but also something from which the term follows. In the second meaning, analysis is a regression whereby one posits what is sought, say, A, and tries to find a B_1 that implies A, and then a B_2 that implies B_1, until one attains a final B_f that is a "thing that is already known or occupies the rank of a principle." The synthesis would be a deduction of A from B_f.

The third paragraph, however, suggests that also the first meaning is to be taken into account: the analysis would be a deduction of B_f from A (say, if A then B_1, if B_1 then B_2, if B_2 then B_f). If B_f is "something established to be false" or "impossible," by *modus tollens*, one deduces that A is false or impossible, too. If, instead, B_f is "something established to be true" or "possible," one needs to reverse the steps of the analysis in order to achieve a deduction of A from B_f. Reversing the steps is, of course, possible only if the analytical deduction is constituted of biconditionals (of the form "if and only if"): this requirement is often satisfied in geometrical constructions.

The ambiguity in Pappus's characterization of the method of analysis went hand in hand with the ambiguities that one encountered when inspecting examples of its application in the *Collectio*. It was very difficult to extract clear indications from Pappus's mathematical practice. Early-modern mathematicians had to confront a mathematical methodology that eluded a clear definition.

Another distinction of momentous importance for early-modern mathematicians is that between problems and theorems. A problem calls for a construction, achieved via permitted means, for its solution. It starts from certain elements considered as already constructed either by postulate or by previous constructions. A problem ends with a Q.E.I. (quod erat inveniendum, what was to be found) or with a Q.E.F. (quod erat faciendum, what was to be done). A theorem asks for a deductive proof, a sequence of propositions one following from the previous one by permitted inference rules. The starting point of the deductive chain can be either axioms or previously proved theorems. A theorem ends with Q.E.D. (quod erat demonstrandum, what was to be demonstrated). According to Pappus, therefore, there are two kinds of analysis, the former referring to theorems, the latter to problems. But it is clear that early-modern mathematicians were mainly concerned with the analysis of geometrical problems.

The nature of the method of analysis and synthesis (or resolution and composition) in ancient Greek mathematics was not, and still is not, completely clear. As Pappus was the main source of inspiration both for Descartes and for Newton, it is appropriate to look at one of the demonstrations from the *Collectio*— Proposition 48, Book 4—which can be taken as an example of application of problematic analysis and synthesis (figure 3.1):

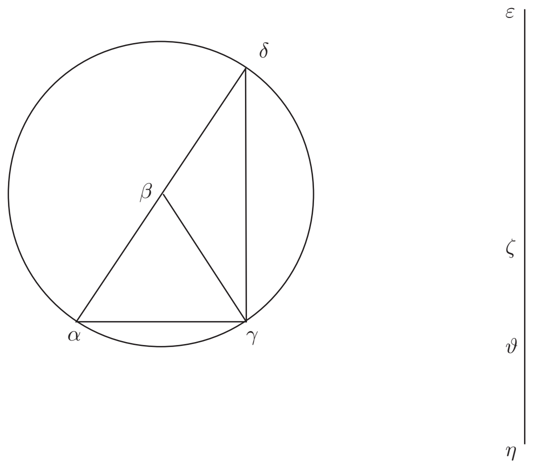


Figure 3.1
Proposition 48, Book 4. Source: Pappus, *Collectionis Quae Supersunt* (1876–78), 1, p. 291. Courtesy of the Biblioteca Angelo Mai (Bergamo).

> Prop. 48. To construct an isosceles triangle having each of the angles at the base with a given ratio to the remaining angle.
>
> [analysis] Let it have happened, and let $\alpha\beta\gamma$ be constructed, and let circle $\alpha\delta\gamma$ be drawn about center β and through points α, γ, and let $\alpha\beta$ be extended to δ, and let $\delta\gamma$ be joined. Since, then, the ratio of angle $\gamma\alpha\beta$ to angle $\alpha\beta\gamma$ is given, and the angle at δ is half the angle $\alpha\beta\gamma$, the ratio of angle $\gamma\alpha\delta$ to angle $\alpha\delta\gamma$ is therefore given, in such a way that so is the ratio of arc $\delta\gamma$ to arc $\alpha\gamma$. And so, since arc $\alpha\gamma\delta$ of the semicircle is cut in a given ratio, γ is given, and triangle $\alpha\beta\gamma$ is given in species.
>
> [synthesis] It will be synthesized as follows. For let the given ratio that each of the angles at the base must have to the other angle be the ratio of $\epsilon\zeta$ to $\zeta\eta$, and let $\zeta\eta$ be bisected at θ, and let a circle $\alpha\delta\gamma$ be set out with center β and diameter $\alpha\delta$, and let arc $\alpha\gamma\delta$ be cut at γ, in such a way that as arc $\delta\gamma$ is to arc $\gamma\alpha$, so is $\epsilon\zeta$ to $\zeta\eta$ (for this was previously described, i.e., how generally a given arc may be cut in a given ratio), and let $\beta\gamma$, $\gamma\alpha$, $\gamma\delta$ be joined. Thus, since arc $\delta\gamma$ is to arc $\gamma\alpha$, i.e., as angle $\delta\alpha\gamma$ to $\alpha\delta\gamma$, so is $\epsilon\zeta$ to $\zeta\theta$; and twice the consequents, therefore, as angle $\gamma\alpha\beta$ is to angle $\alpha\beta\gamma$, so is $\epsilon\zeta$ to $\zeta\eta$. Therefore an isosceles triangle, $\alpha\beta\gamma$, is constructed having each of the angles at the base having a given ratio to the remaining angle.[8]

The analysis begins with the supposition that the isosceles triangle has been constructed. The deduction leads to the discovery of certain identities between ratios that must hold under the supposition that the required construction has

[8] Pappus, *Collectionis Quae Supersunt* (1876–78), 1, pp. 288.15–290.23. I thank Henry Mendell for providing this translation from the Greek, and Fabio Acerbi for further assistance.

been achieved. These identities allow us to determine the triangle "in species," that is, they determine a class of similar isosceles triangles that answer the conditions of the problem. In the synthesis, the triangle that answers the problem is instead constructed starting from the identities between ratios that are assumed in the statement of the problem, or that have already been achieved in previous constructions.

It should be noted that one of the previously proved presuppositions deployed in the analysis and synthesis of Proposition 48 falls outside the canon of ruler and compass constructions, which was often identified with the canon allowed in Euclid's *Elements*, namely, Pappus assumes as a "thing which is already established" that any arc can be cut in two arcs standing in have a given ratio. The proof is in Propositions 45 and 46, Book 4, and is achieved by the use of the "symptoms" (the characteristic properties) of the quadratrix and the spiral, respectively.[9]

A powerful idea that began to circulate at the turn of the seventeenth century was that modern symbolic algebra captured some aspects of the analysis of the Greeks. The evidence that a method analogous to symbolic algebra was within reach of the ancients was provided by the work of Diophantus and especially by an ahistorical reading of Euclid's *Elements* and *Data*. The approach of Renaissance culture toward the classics, in sculpture, architecture, music, philosophy, and other fields, was characterized by admiration together with a desire to restore the forgotten conquests of the ancients. This approach, often bordering on worship, carried with it the idea that there had been a decay after a glorious, golden past. The works of Euclid, Apollonius, and Archimedes were considered by many Renaissance mathematicians to be unsurpassable models. The question that often emerged was, How could the Greeks have achieved such a wealth of results?

In the decades following the publication of the *Collectio* the belief in the existence of a lost or hidden "Treasure of Analysis" promoted many efforts aimed at restoring the ancients' method of discovery. Not everybody trod in the steps of the classicists. Typically, some promoters of the new symbolic algebra were proud to define themselves as innovators rather than as restorers. It was common, however, even among creative algebraists such as François Viète, John Wallis, and Isaac Newton, to relate symbolic algebra to the ancient analysis, to the hidden problem-solving techniques of the ancients. Viète's main work, significantly entitled *In Artem Analyticem Isagoge* (an introduction to the analytical art), published in 1591, opens with reference to the ancients' knowledge of analysis.

Reference to the remote past has often been used (e.g., by Copernicus in *De Revolutionibus* (1543)) to validate theories that appear to us extremely innovative. It is always a difficult historiographic matter to evaluate the rhetorical role of such

[9] Pappus, *Collectionis Quae Supersunt* (1876–78), 1, pp. 285–9.

declarations, for instance, was Viète really convinced that he was a rediscoverer of past methods, or was he merely using the authority of the ancients in order to render new ideas acceptable? It is often the case that references to the lost ancient tradition played different roles in an author's several works. The identification of algebra with the analytical methods of discovery of the ancients became a rather common theme such that it could be found in a widely circulated seventeenth-century mathematical dictionary by Jacques Ozanam (1691).[10] However, as the seventeenth century progressed, the tumultuous advance of mathematical techniques rendered references to the ancients more and more strained. What mathematicians could obtain appeared to many obviously not within the grasp of the ancient geometers. Thus, Newton found himself trapped in this tension between ancient and modern.

3.2 Analysis and Synthesis in Descartes' Mathematical Canon

3.2.1 Algebra as analysis

How did Descartes define his canon of problem solving and the role of algebra in the analysis and synthesis of geometrical problems? The historian who has done the most to clarify this question is Bos.[11]

In Book 1 of the *Géométrie*, Descartes explained how one can translate a geometrical problem into an equation.[12] He was able to do so by a revolutionary departure from tradition. Indeed, he interpreted algebraic operations as closed operations on segments. For instance, if a and b are segments, the product ab was not conceived by Descartes as representing an area but rather another segment: "It must be observed that by a^2, b^3, and similar expressions, I ordinarily mean any simple lines." Before the *Géométrie* the multiplication of two segments of lengths a and b would have been taken as a representation for the area of a rectangle with sides measuring a and b.[13]

Descartes' interpretation of algebraic operations was a gigantic innovation (figure 3.2),[14] but he proceeded wholly in line with Pappus's method of analysis and synthesis, which starts from the assumption that the problem is solved. Indeed, according to Descartes, one has to "start by assuming that the problem was solved

---

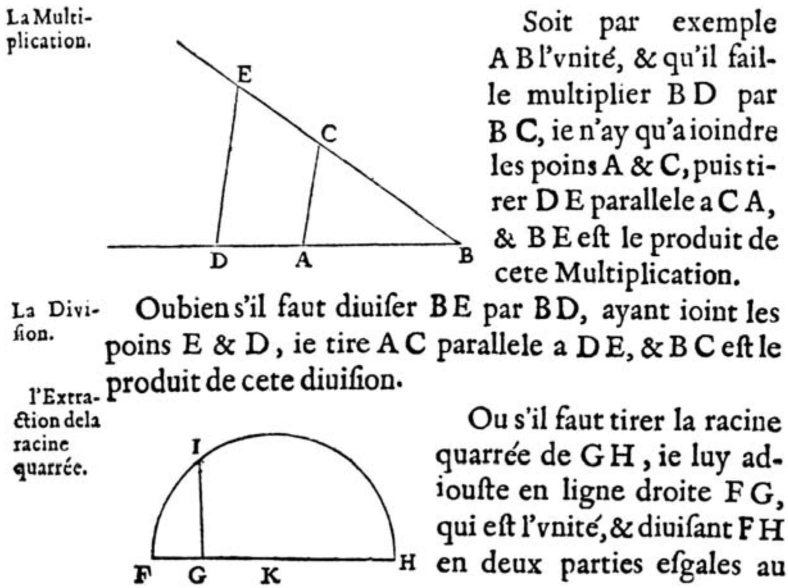
[10] Ozanam, *Dictionaire Mathématique* (1691).

[11] Bos's reading of Descartes is recognized as a seminal contribution. Nonetheless, objections have been raised to the close association that he establishes in *Redefining Geometrical Exactness* (2001) between Descartes' mathematical work and the exigencies emerging from the tradition of the analysis and synthesis of geometrical problems.

[12] Descartes did so in the sixth section of Book 1, entitled "Comment il faut venir aux Equations qui servent a resoudre les problesmes" [How one should arrive at the equations that serve for solving the problems]. *Géométrie*, p. 300 [8].

[13] *Géométrie*, p. 298 [5].

[14] Were his contemporaries and indeed Descartes himself aware of this?



La Multi-
plication.

Soit par exemple A B l'vnité, & qu'il faille multiplier B D par B C, ie n'ay qu'a ioindre les poins A & C, puis tirer D E parallele a C A, & B E eſt le produit de cete Multiplication.

La Diui-
ſion.

Oubien s'il faut diuiſer B E par B D, ayant ioint les poins E & D, ie tire A C parallele a D E, & B C eſt le produit de cete diuiſion.

l'Extra-
ction dela
racine
quarrée.

Ou s'il faut tirer la racine quarrée de G H, ie luy adiouſte en ligne droite F G, qui eſt l'vnité, & diuiſant F H en deux parties eſgales au

Figure 3.2

Descartes' geometrical interpretation of algebraic operations. He wrote: "For example, let AB be taken as unity, and let it be required to multiply BD by BC. I have only to join the points A and C, and draw DE parallel to CA; and then BE is the product of BD and BC." So, given a unit segment, the product of two segments is represented by another segment, not by a surface. The second diagram is the construction of the square root of GH. Given GH and a unit segment FG, one draws the circle of diameter $FG + GH$ and erects GI, the required root. Source: Descartes, *Géométrie* (1637), p. 298. Courtesy of the Biblioteca Angelo Mai (Bergamo).

and consider a figure incorporating the solution."[15] The line segments in the figure are denoted by letters, a, b, c, ... for segments that are given, and z, y, x, ... for segments that are unknown. Geometrical relations holding between the segments (which in general take the form of identities between ratios of lengths but can also be more complicated relations such as cross-ratio identities) are then translated into corresponding equations.[16] Here we are at the very beginning of the analytical process: the unknown segments are treated as if they were known and manipulated in the equations on a par with the givens of the problem. This is why Descartes and the others of the analytical school associated algebra with the method of analysis. Algebra was not associated with the method of analysis because of the use of symbolism (we are accustomed nowadays to equate analytical with symbolical, a

[15] Bos, *Redefining Geometrical Exactness* (2001), p. 303.

[16] The concept of cross-ratio is defined in §5.2.2.

linguistic convention that began to surface in the middle of the eighteenth century) but rather because algebra was perceived as patterned on the method of analysis described by Pappus. This is a very important point. A quotation from Descartes will clarify it:

> If, then, we wish to resolve any problem, we first suppose the solution already effected, and give names to all the lines that seem needful for its construction, to those that are unknown as well as to those that are known. Then, making no distinction between known and unknown lines, we must unravel the difficulty in any way that shows most naturally the relations between these lines, until we find it possible to express a single quantity in two ways. This will constitute an equation. . . . This is one thing which I believe the ancient mathematicians did not observe, for otherwise they would not have put so much labor into writing so many books in which the very sequence of the propositions shows that they did not have a sure method of finding all, but rather gathered together those propositions on which they had happened by accident.[17]

As is apparent from this quotation, the algebraic symbolism and the formulation of equations are, according to Descartes, the tools to be applied in the analysis of geometrical problems. He proposed a great number of examples that displayed how successful this analytical procedure could be. He taught how geometrical problems that had given trouble to Greek mathematicians could be easily translated into algebra. Algebra was the new analytical tool that, in his opinion, allowed one to surpass the boundaries of ancient Greek geometry. It is difficult for us to appreciate how irreverent such a position would have seemed to the heirs of the Renaissance.

The geometrical problems that Descartes considered can be divided into two classes: determinate and indeterminate.

3.2.2 Analysis and Synthesis of Determinate Problems

As Bos has explained, Descartes' method of problem solving was, according to the Pappusian canon, divided into an analytical and a synthetic part.[18] The analytical part was algebraic and consisted in reducing the problem to a polynomial equation.

[17] "Ainsi voulant resoudre quelque problesme, on doit d'abord le considerer comme desia fait, & donner des noms a toutes les lignes, qui semblent necessaires pour le construire, aussy bien a celles qui sont inconnües, qu'aux autres. Puis sans considerer aucune difference entre ces lignes connües, & inconnües, on doit parcourir la difficulté, selon l'ordre qui monstre le plus naturellement de tous en qu'elle sorte elles dependent mutuellement les unes des autres, iusques a ce qu'on ait trouvé moyen d'exprimer une mesme quantité en deux façons: ce qui se nomme une Equation . . . . Ce que ie ne croy pas que les anciens ayent remarqué. Car autrement ils n'eussent pas pris la peine d'en escrire tant des gros livres, ou le seul ordre de leurs propositions nous fait connoistre qu'ils n'ont point eu la vraye methode pour les trouver toutes, mais qu'ils ont seulement ramassé celles qu'ils ont rencontrées." *Géométrie*, pp. 300 [6–9], 304 [17].

[18] Bos, *Redefining Geometrical Exactness* (2001), pp. 287–9.

Table 3.1 Descartes on Analysis and Synthesis of Determinate Problems

Required object	Line segments
Analysis	Express algebraically ratios of known and unknown segments until an equation is achieved
Equation	Algebraic in one unknown
Synthesis	"Construction of the equation": determine line segments representing the roots of the equation by intersection of algebraic curves of lowest possible degree

If the equation was in one unknown, the problem was determinate. The equation's real roots would correspond to the finitely many solutions of the problem. Methods for the calculation of the roots of algebraic equations up to the fourth degree had been achieved in the sixteenth century. No general formulas for equations of degree greater than 4 were known, however. At the beginning of the nineteenth century, it was Évariste Galois (and Niels H. Abel) who proved that they do not exist. But even when formulas were available, they did not provide indications about how one could achieve what was sought: a geometrical construction. In some cases, the formulas, such as those of Girolamo Cardano and Ludovico Ferrari, involved square roots of negative quantities, which at the time did not possess a geometrical interpretation. Strange as it might appear to us, algebra could do only half of the business required by early-modern mathematicians; a geometrical construction was needed.

The analysis, or resolution, was not, according to early-modern standards, the solution of the problem. The solution of the problem must be a geometrical construction of what is sought in terms of legitimate geometrical operations (Q.E.F.). The resolution belongs to the analytical stage of the problem-solving canon. Descartes employed algebra as the tool for problematic analysis. But analysis must be followed by synthesis; the resolution must be followed by composition (note the double Greek and Latin terminology). We now have to move back from algebra to geometry. After Descartes, the synthetic process was known as the construction of the equation (table 3.1). Descartes accepted from the tradition the idea that such constructions must be performed by intersection of curves.

The construction of the equation presented the geometer with a new problem, not always an easy one. One had to choose two curves such that their intersections determine segments that are the solutions of the problem. These segments are the construction required by the problem. The fact that their lengths geometrically represent the finitely many real roots of the equation can be noted, but it played a secondary role in Descartes' canon. Generally such a construction by intersection of curves was followed by a geometrical proof that the problem had been solved as required.

The synthetic part of the process of problem solving opened up a series of questions. Which curves are admissible in the solution of problems? When can they be considered as known or given? Which curves, among the admissible and constructible, are to be preferred in terms of simplicity? In asking himself these questions Descartes was continuing, albeit on a different plane of abstraction and generality, a long debate concerning the role and classification of curves in the solution of problems, an ancient tradition that was transmitted to early-modern mathematicians such as Viète, Marino Ghetaldi (Getaldić), Johannes Kepler, and Fermat by way of Pappus.[19] Descartes prescribed that in the construction of the equation one had to use algebraic curves of lowest possible degree.

3.2.3 Demarcation and Simplicity

The demarcation between admissible and nonadmissible curves, and the classification of the admissible ones in Descartes' *Géométrie*, are well-known topics in the history of seventeenth-century mathematics.[20] Descartes began Book 2, devoted to the nature of curved lines, by excluding mechanical curves from the scope of geometry: they could not be used as means of constructions. Mechanical curves are what we would nowadays call transcendental curves.[21] Descartes mentions only two examples: the (Archimedean) spiral and the quadratrix (in his commentary to the Latin translation of the *Géométrie*, van Schooten added the cycloid). All the other *plane* curves about which he knew were geometrical (we would call them algebraic) and could be admitted into geometry.

Descartes continued by criticizing the classification of problems into planar, solid, and linear, a classification that, according to Pappus, was adopted by the ancients.[22] Planar problems are those that can be constructed by using circles and straight lines, whereas solid problems cannot be solved by ruler and compass but can be constructed by means of conic sections. Linear problems require the use of more complex (*plus composées*) curves. This classification does not fit with Descartes' innovative program. To put it briefly (perhaps too simplistically): he accepted only curves defined by a polynomial equation in two unknowns (an algebraic equation) and classified them in terms of the degree of their defining equation.[23] In any case,

---

[19] Bos, *Redefining Geometrical Exactness* (2001), pp. 287–9.

[20] Rashed, "La *Géométrie* de Descartes et la Distinction Entre Courbes Géométriques et Courbes Mécaniques" (1997).

[21] The two concepts are extensionally equivalent, but a historian should nevertheless be careful not to conflate them.

[22] It is unclear from Pappus's text whether he approved this classification.

[23] It seems that Descartes as well as other seventeenth-century mathematicians assumed as a matter of course that radicals can always be removed from an equation by reordering the terms and raising to a suitable power. Bos, "Arguments on Motivation in the Rise and Decline of a Mathematical Theory" (1984), p. 339.

this is the message that Newton derived from the *Géométrie* and that he repeatedly attributed to Descartes. Thus, the conic sections belong to the same class since, given a system of rectangular coordinates, they are the loci of points satisfying second-degree algebraic equations.[24] By contrast, according to Descartes' reading of Pappus, the ancients classified the spiral, the quadratrix, the conchoid, and the cissoid as more complex curves. The cissoid was often used for finding two mean proportionals (and thus, for instance, for solving the problem of the duplication of the cube), the conchoid for neusis constructions (classic applications were the trisection of the angle and the duplication of the cube); all these were solid problems that could be solved by intersection of conics. The spiral and the quadratrix could be used for linear problems (such as general angular sections or the quadrature of the circle). However, the conchoid and the cissoid are loci of algebraic equations; thus, according to Descartes, they may be admitted into geometry, being placed in a higher-order class relative to the conics because their equations have degree greater than 2. The spiral and the quadratrix, on the other hand, have no algebraic defining equation.

The category of exactness enters prominently into Descartes' characterization of geometry as opposed to mechanics: "[W]e make the usual assumption that geometry is precise and exact, while mechanics is not."[25]

The spiral and quadratrix cannot be admitted into geometry because they lack exactness, they are mechanical curves. Why do they lack exactness? Descartes' answer is that these curves are described by two motions whose relation (*raport*) cannot be measured exactly:

> [T]he spiral, the quadratrix, and similar curves, which really do belong only to mechanics, are not among those curves that I think should be included here, since they must be conceived of as described by two separate movements whose relation does not admit of exact determination.[26]

---

[24] There is some ambiguity in Descartes concerning the relation between the circle and the other conic sections. Algebraically speaking, the circle would be classified with the conics. However, Descartes was aware that the circle had constructionally less power. Newton seized the opportunity of criticizing Descartes on this point (see chapter 4).

[25] "prenant comme on fait pour Geometrique ce qui est precis & exact, & pour Mechanique ce qui ne l'est pas." *Géométrie*, p. 316 [43]. I quote from van Schooten's Latin translation of the *Géométrie* because this is the edition that Newton used: "Geometricum censeamus illud, (ut fieri solet) quod omnino perfectum atque exactum est, & Mechanicum quod ejusmodi non existit." *Geometria*, p. 18. This was commonplace in the seventeenth century. For example, Wallis in his *Mechanica* (1670–1) wrote, "In re Geometrica; *Mechanice* quid factum, non Geometrice, dici solet; quando rudi $\chi\epsilon\iota\rho\upsilon\rho\gamma\iota\alpha$, vel materialis instrumenti applicatione, aliisve mediis non absimilibus, aliquid metimur: non $\alpha\pi o\delta\epsilon\iota\kappa\tau\iota\kappa\omega\varsigma$." Wallis, *Opera*, 1, p. 575.

[26] "la Spirale, la Quadratrice, & semblables, qui n'appartienent veritablement qu'aux Mechaniques, & ne sont point du nombre de celles que je pense devoir icy estre recevues, a cause qu'on les imagine descrites par deux mouvemens separés, & qui n'ont entre eux aucun raport qu'on puisse mesurer exactement." *Géométrie*, p. 317 [44] = "Spiralis, Quadratrix, atque similes; quae

This and similar passages of the *Géométrie* devoted to the characterization of mechanical curves are far from clear. In these lines, Descartes affirms that mechanical curves are generated by two separate motions that have no exact measurable relation. For instance, the Archimedean spiral can be generated by two motions: the rotary motion of a ruler that turns with constant angular speed around a pivot; and the rectilinear motion of a point P that slides with constant speed along the ruler. Point P traces a spiral. By tuning the rates of increase of the two separate motions, one generates different spirals. Descartes does not admit curves generated by such constructions into the framework of the *Géométrie*, since they are "described by two separate movements, between which there is no relation that can be measured exactly." As Bos has argued, this statement can be understood to mean that "[i]n view of the methods of tracing ... the spiral, we may conclude that when Descartes spoke about the measures of the motions, he meant their velocities. Indeed, these measures have no exact measurable *raport*, as the comparison of the velocities involves the comparison of the lengths of straight and curved lines."[27]

The example of the spiral can clarify this matter. As noted, one can generate any Archimedean spiral by imparting suitably chosen radial and angular motions to a point. For instance, one can require that the angular motion make one complete revolution in the time required for the rectilinear motion to traverse a given distance. The ratio of the two component velocities involves the ratio of the lengths of straight and curved lines. But according to Descartes, this ratio can be known only in an approximate way, and one cannot count it as an exactly measurable *raport*. As Descartes remarks later in the *Géométrie*, "[T]he ratios between straight and curved lines are not known, and I believe cannot be discovered by human minds."[28]

In this respect, Bos argues, Descartes could not simply define geometrical curves as those that have a defining algebraic equation and mechanical curves as those that do not have such a defining equation. Bos wrote, "[B]ecause Descartes did not consider the equation a sufficient representation of the curve, he could not establish any distinction between geometrical and non-geometrical [mechanical] curves on the basis of their equations; he had to reason about it on the basis of representations of curves which he did find acceptable."[29] Descartes devoted much of Book 2 to very complex and fascinating reasoning concerning curves, their representations (by point-wise construction and by tracing machinery involving linked rulers or strings),

revera non nisi ad Mechanicas pertinent, nec ex illarum numero sunt, quas hic recipiendas autumo: quandoquidem illas duobus motibus describi imaginamur, qui a se invicem sunt diversi, nec ullam inter se relationem habent, quae exacte mensurari possit." *Geometria*, pp. 18–9.

[27] Bos, *Redefining Geometrical Exactness* (2001), p. 314.

[28] "la proportion, qui est entre les droites & les courbes, n'estant pas connuë, & mesme ie croy ne le pouvant estre par les hommes." *Géométrie*, p. 340 [91].

[29] Bos, *Redefining Geometrical Exactness* (2001), p. 297.

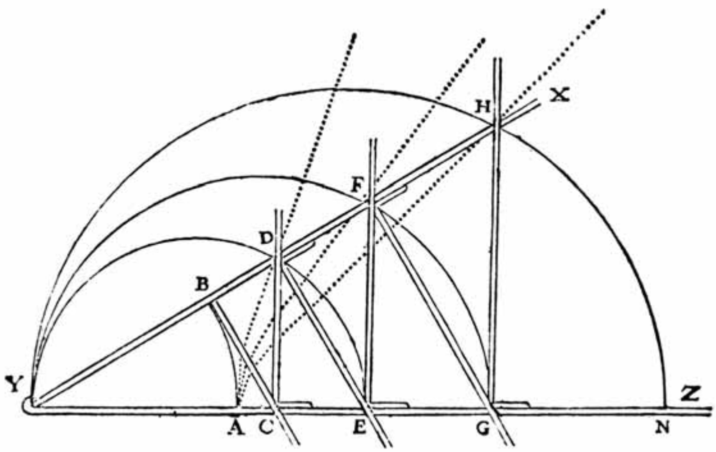


Figure 3.3
Descartes' mesolabum. The arrangements of sliding rulers is such that the points D, F, and H, when the angle XYZ increases from 0, describe the dotted curves. If $YA = YB = a$, $AC = x$, $CD = y$, then $x^4 = a^2(x^2 + y^2)$. The curves traced by F, H, etc. have equations $x^8 = a^2(x^2 + y^2)^3$, $x^{12} = a^2(x^2 + y^2)^5$, etc. Source: Descartes, *Géométrie* (1637), p. 318. Courtesy of the Biblioteca Angelo Mai (Bergamo).

and the conditions that guarantee the existence of an algebraic equation defining them. His effort was to isolate those constructions that give rise to curves defined by algebraic equations. A famous example is his mesolabum (figure 3.3).

Before proceeding with this brief presentation of Descartes' canon of problem solving, it seems necessary to ask why Descartes had to define curves by tracing mechanisms rather than by relying on their defining equations. In order to answer this question, one must remember that curves enter the *Géométrie* not only as loci that answer a problem but also as devices for constructing points by intersection. Now, algebra did provide Descartes with a method for constructing curves, but not a method that yielded exact results. From an algebraic equation $f(x, y) = 0$, Descartes could obtain a point-wise construction of the curve. That is, he indicated that one could choose a sequence of values of one variable, say x_1, x_2, x_3, ..., and determine via the equation the corresponding values of the other variable y_1, y_2, y_3, ...[30] In general, one will obtain a representation of the curve as a set of

[30] Note that this procedure is not without problems because the equations for the separate y_is are equations in one unknown, but with probably any degree, and their coefficients are different

points in the plane, not the result one would desire when looking this way for the intersection of two curves. If, on the other hand, one has a tracing mechanism that generates the curves by continuous motion, points of intersection can be thought of as determined exactly. Since antiquity, devices for tracing curves by instruments, moving rulers or strings, were known; this was a chapter of geometry that went under the rubric organic [from $o\rho\gamma\alpha\nu o\nu$, instrument] description of curves. Organic descriptions were known for the straight line and circle (ruler and compass), for the conic sections, for the conchoid, for the spiral, and for the quadratrix. The cissoid was defined point-wise.[31]

What counts here is that Descartes, while requiring organic descriptions of geometrical curves in order to determine their intersections exactly, rejected a number of curves as mechanical. He claimed that they are not certain and exact, that they are not the loci of a finite algebraic equation in two unknowns, that they are described by two motions (velocities) with no relation that can be measured exactly, since this relation implies the determination of π.[32] As van Schooten wrote, simplifying slightly in his commentary to Book 2 of the *Géométrie*, the curves to be rejected from geometry are defined as follows:

> It is then moreover easy to understand, which are those [curves] that are to be repudiated from Geometry, and to be put amongst the Mechanical: ... these are all the curves which cannot be described by a continuous motion, but which can be conceived of as described by two motions, which are separate one from the other, and which have no relationship, which can be measured exactly; or those whose points have no relation, expressible by means of an equation common to all, to the points of a straight line.[33]

At the time of the publication of the *Géométrie*, the rejection of inexact mechanical curves implied a very limited loss; in 1637 mathematicians could reckon amongst the plane curves only the quadratrix, the Archimedean spiral, the logarithmic curve, and the cycloid as mechanical.[34]

---

for different x_is.

[31] Newton devised an organic description of the cissoid (§4.5).

[32] That is why, in the *Principia*, Newton calls geometrical curves "geometrically rational" and mechanical curves "geometrically irrational" (§13.3).

[33] "Ubi porro facile est intelligere, quaenam sint, quae ex Geometria sint rejiciendae & inter Mechanicas ponendae: ... sunt illae omnes, quae per motus continuos describi nequeunt, ... sed per duos motus describi concipiuntur, qui sunt a se invicem distincti, nullamque relationem habentes, quae possit exacte mensurari, sive quarum omnia puncta ad omnia lineae rectae puncta relationem non habent, quae per aliquam aequationem omnibus communem exprimi possit." *Geometria*, p. 167.

[34] In his correspondence Descartes showed an interest in mechanical curves that is not evident in the *Géométrie*.

By contrast, a geometric curve, in Descartes' terminology, can be conceived of as described by two motions whose relation is measured exactly. Descartes claimed that such curves are expressed by a finite algebraic equation:

> [All] points of those curves which we may call geometric, that is, those which admit of precise and exact measurement, must bear a definite relation to all points of a straight line, and that this relation must be expressed by means of a single equation. If this equation contains no term of higher degree than the rectangle of two unknown quantities, or the square of one, the curve belongs to the first and the simplest class, which contains only the circle, the parabola, the hyperbola and the ellipse.[35]

One can recognize here the definition of a conic as a curve defined by an equation of the form $x^2 + ay^2 + bxy + cx + dy + e = 0$.[36]

Note an important point concerning Descartes' methodology. Descartes devoted many pages of the *Géométrie* to various constructions of curves by tracing apparatuses. One of his purposes was to exclude illegitimate constructions that generate mechanical curves. However, he carried out the classification of legitimate, geometrical curves exclusively in terms of their algebraic equations. According to Descartes, it is indeed the degree of the equation that defines the simplicity of a geometrical curve. Thus, in the end, Descartes prescribed his golden rule for the solution of determinate problems: in constructing the equation one must choose from among the admissible (i.e., geometrical) curves those that are loci of polynomial equations of lowest degree.

3.2.4 Analysis and Synthesis of Indeterminate Problems

In some cases, the algebraic analysis of a problem does not lead to an equation in one unknown (as when the problem is determinate) but to an equation in two or more unknowns. In this case, the problem is indeterminate because it admits an infinity of solutions. Typically, Descartes considered problems that were reduced

---

[35] "tous les poins, de celles qu'on peut nommer Geometriques, c'est a dire qui tombent sous quelque mesure précise & exacte, ont necessairement quelque rapport a tous les poins d'une ligne droite, qui peut estre exprimé par quelque equation, en tous par une mesme, Et que lorsque cete equation ne monte que iusques au rectangle de deux quantités indeterminées, oubien au quarré d'une mesme, la ligne courbe est du premier & plus simple genre, dans lequel il ny a que le cercle, la parabole, l'hyperbole, & l'Ellipse qui soient comprises." *Géométrie*, p. 319 [48] = "puncta omnia illarum [of the curves], quae Geometricae appellari possunt, hoc est quae sub mensuram aliquam certam & exactam cadunt, necessario ad puncta omnia lineae rectae, certam quandam relationem habeant, quae per aequationem aliquam, omnia puncta respicientem, exprimi possit. Et quod, cum aequatio haec non ultra ractangulum duarum quantitatum indeterminatarum, aut non ultra quadratum unius ex illis ascendit, linea curva tunc primi & simplicissimi sit generis; (sub quo tantum Circulus, Parabola, Hyperbola, & Ellipsis sunt comprehensae:)" *Geometria*, p. 21.

[36] Descartes was aware of the fact that all conics are expressed by second-degree equations. He was also aware of cases when the conic degenerates into a pair of straight lines.

Table 3.2 Descartes on analysis and synthesis of indeterminate problems

Required object	Curve
Analysis	Express algebraically ratios of known and unknown segments until an equation is achieved
Equation	Algebraic in two unknowns
Synthesis	"Construction of the curve": (i) point-wise, (ii) (if second degree) use Apollonius's *Conics*, (iii) motion of curves, (iv) organic

to polynomial equations in two unknowns. Their infinitely many solutions form a one-dimensional locus in a plane: a curve or a straight line.[37] These so-called locus problems (problems whose solution is achieved with the construction of a geometrical locus, often a plane curve) formed an important part—indeed, the higher, more complex part—of the ancient tradition of problem solving. Descartes proposed algebra as the analytical tool for locus problems; he reduced them to equations. The equation, however, did not constitute the solution of the problem; a geometrical construction was required.

But how one should construct a curve? Descartes' answer was less straightforward compared with the canon of synthesis for determinate problems (see table 3.2). In Book 1 of the *Géométrie* he intimated that curves should be constructed point-wise by choosing arbitrary values for one unknown. In this way, one could obtain a succession of arbitrarily many points on the curve. As noted in §3.2.3, this procedure does not allow one to use curves as a means of construction, since in this case one cannot determine the intersection of two curves exactly. Further, point-wise constructions, for instance, Christoph Clavius's proposed construction of the quadratrix, were often criticized as lacking in exactness.

The *Géométrie* contains other approaches to curve construction. One is based on the theory of conics. In Book 2, Descartes showed how the coefficients of a second-degree equation could be used to determine geometrical parameters that define a conic according to the theory developed by Apollonius. The Apollonian construction is preferable to a point-wise construction because the conic is given in its entirety and not at isolated points, but it postulates the possibility of cutting a cone in a prescribed inclination with a plane, a postulate that appeared to many far too complex.

Another alternative was intended to generate the curve by motion. Descartes conceived cases in which curves intersect and move in prescribed ways such that

[37] Bos, *Redefining Geometrical Exactness* (2001), pp. 310–11.

their intersection traces a new curve. He devised also tracing mechanisms involving linked rulers (figure 3.3) and threads (figure 3.4).

Descartes devoted great efforts to evaluating and relating these different methods of curve construction one to the other, and to studying their relation to the algebraic representation of curves. His purpose was to isolate constructions that generate loci of algebraic equations in two unknowns. It is beyond my present purposes to discuss Descartes' ideas on this topic, and I refer the reader to Bos's work, to which I am deeply indebted in this chapter.[38] Newton rejected all the means of curve construction proposed by Descartes (see chapter 5).

3.3 An Example of Analysis and Synthesis of a Determinate Problem

Having discussed Descartes' canon in broad outlines, I now present some examples that show the canon at work: first a determinate problem that Descartes proposed in the *Géométrie*, the classic problem of angle trisection; and then, in the next section, an indeterminate problem, the Pappus problem, which was to play an important role in Newton's confrontation with the Cartesian canon (see chapter 5).

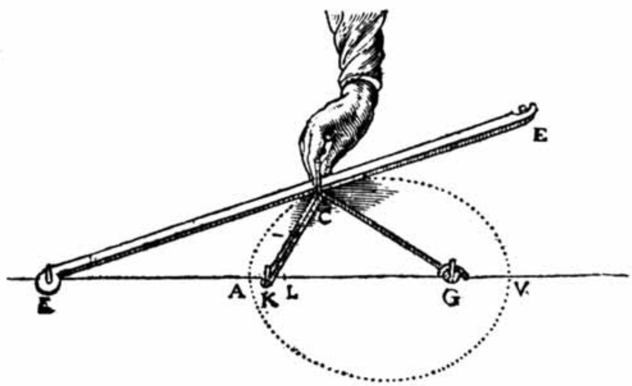


Figure 3.4

Construction of a Cartesian oval with ruler, pencil, and thread. The oval reproduced is the curve consisting of all those points C for which the sum of the distance to one focus G plus twice the distance to a second focus K is a constant. Descartes constructed this oval using two pins, a thread, a pencil, and a ruler. The thread "being attached at E to the end of the ruler, passes from C to K and then back to C and from C to G, where its other end is fastened." Source: Descartes, *Géométrie* (1637), p. 356. Courtesy of the Biblioteca Angelo Mai (Bergamo).

[38] See especially Bos, *Redefining Geometrical Exactness* (2001), pp. 225 ff.

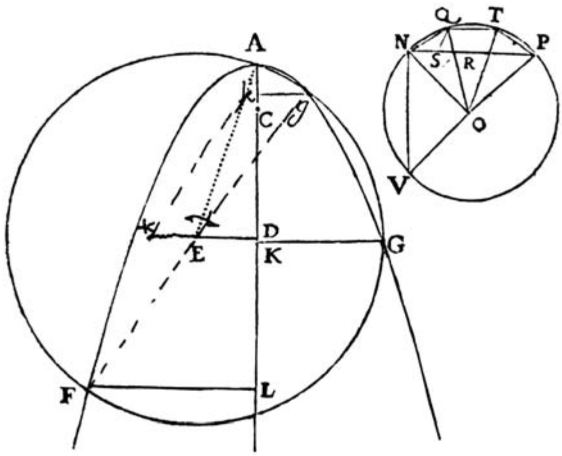


Figure 3.5
Construction of a third-degree equation in Descartes' *Géométrie*. The problem of trisecting the angle NOP is *resolved* (resolutio is the Latin translation of the Greek analysis) by a third-degree equation. Descartes *constructed* the equation (constructio or compositio translate synthesis) via intersection of circle and parabola. The segments kg, KG, and LF represent two positive and one negative root. The smaller of the two positive roots kg must be "taken as the length of the required line NQ." Source: Descartes, *Géométrie* (1637), p. 396. Courtesy of the Biblioteca Angelo Mai (Bergamo).

3.3.1 Statement of the Problem

Let it be required to divide the angle NOP (figure 3.5), or rather, the circular arc NQTP, into three equal parts.[39]

3.3.2 Analysis: Invention of the Equation

The first steps of the canon constitute the analysis of the problem, and algebra is the tool for geometrical analysis.

1. Given a geometrical problem, start by assuming that the required geometrical construction has been already effected, and draw a corresponding figure.
2. Name all segments in the figure by letters: those known by a, b, c, etc., and those unknown by z, y, x, etc.
3. Obtain relations between the letters and manipulate them until some standard form (in this case a third-degree equation in one unknown) is reached.

---

[39] *Géométrie*, p. [207].

This is how Descartes proceeded:

Let $NO = 1$ be the radius of the circle, $NP = q$ be the chord subtending the given arc, and $NQ = z$ be the chord subtending one-third of that arc; then the equation is $z^3 = 3z - q$.

For, drawing NQ, OQ and OT, and drawing QS parallel to TO, it is obvious that NO is to NQ as NQ is to QR as QR is to RS. Since $NO = 1$ and $NQ = z$, then $QR = z^2$ and $RS = z^3$; and since NP or q lacks only RS or z^3 of being three times NQ or z, we have $q = 3z - z^3$.[40]

3.3.3 Synthesis: Construction of the Equation

Descartes employed a general method that he had explained earlier for constructing third- and fourth-degree equations.[41] Given the equation $x^4 = Px^2 + Qx + R$, it is required to construct it (figure 3.6).[42]

1. Describe a parabola with latus rectum equal to 1 and vertex A (its equation is $y = x^2$).
2. Mark D on the y-axis so that $AD = (P + 1)/2$.
3. Draw $DE = Q/2$ horizontally from D in the direction corresponding to its sign.
4. Construct a line segment equal to $\sqrt{\frac{1}{4}(1 + P)^2 + \frac{1}{4}Q^2 + R}$ and draw a circle around E with its radius equal to this line segment.
5. The circle intersects (or touches) the parabola in at most four points $G, F, \ldots$; draw perpendiculars $GK, FL, \ldots$ to the axis from each of these points.
6. The segments $GK, FL, \ldots$, with signs as indicated by their directions, have lengths corresponding to the roots of the equation.

As Bos made clear,

> Descartes proved the correctness of the construction by setting $GK = x$ and calculating the value of the distance EG in two ways, one using that G was on the parabola, the other that G was on the circle; equating both expressions he arrived at the original equation. ... the proof comes down to the following: Put $GK = x$ and $AK = y$, then $y = x^2$ because G is on the parabola. G is also on the circle with center E whose coordinates are $y_E = (P + 1)/2$, $x_E = Q/2$; the equation of the circle is $x^2 - Qx + y^2 - (P + 1)y = R$. Inserting $y = x^2$ one finds from this the equation $x^4 = Px^2 + Qx + R$.[43]

[40] *Géométrie*, p. [207].
[41] This is streamlined following Bos, *Redefining Geometrical Exactness* (2001), pp. 365–6.
[42] Descartes had previously shown how to reduce the equation so that the term in x^3 is removed.
[43] Bos, *Redefining Geometrical Exactness* (2001), p. 366. Descartes did not refer to the equation of the circle here but rather used the Pythagorean theorem.

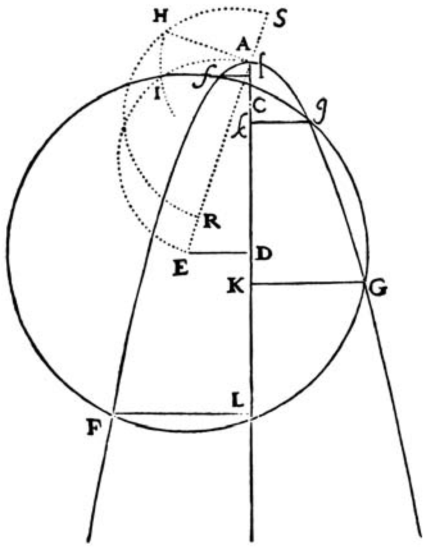


Figure 3.6
Descartes' construction of fourth-degree equations. In this case, the circle cuts the parabola in four points. The ordinates of these points represent two positive and two negative real roots. Source: Descartes, *Géométrie* (1637), p. 392. Courtesy of the Biblioteca Angelo Mai (Bergamo).

These prescriptions apply to a third-degree equation by setting $R = 0$. In this case (figure 3.7), it is easy to show that the circle passes through the vertex A. In the case of angle trisection, Descartes wrote,

> Describe [see figure 3.5] the parabola FAG so that CA, one-half its latus rectum, shall be equal to $1/2$; take $CD = 3/2$ and the perpendicular $DE = q/2$; then describe the circle $FAgG$ about E as center, passing through A. The circle cuts the parabola at three points F, g, and G, besides the vertex A. This shows that the given equation has three roots, namely the two true roots, GK and gk, and one false root, FL.[44]

The smaller segment gk is the geometrical construction (required by the problem) of the line NQ. The problem has now been solved because a geometrical construction of the required chord, which trisects the angle, has been achieved.

[44] A false root is, in Descartes' terminology, a negative root. The other two segments have the following meaning: GK is the chord NV, and (mind the sign) $FL = -(KG + kg)$. *Géométrie*, pp. [207–8].

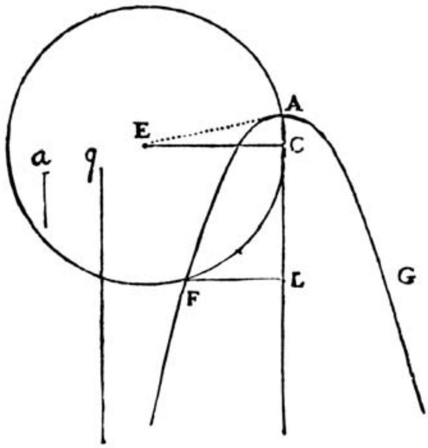


Figure 3.7
Descartes' diagram for the construction of a third-degree equation. Note that Descartes' general construction for third-degree equations requires that the circle cut the parabola's vertex A. Source: Descartes, *Géométrie* (1637), p. 395. Courtesy of the Biblioteca Angelo Mai (Bergamo).

Note that the analytical process (the invention and manipulation of the equation) does not generally give much of a hint about how the constructions can be achieved. The passage from analysis to synthesis is far from being an easy reversal of steps (as Pappus seems to suggest is the case with geometrical analysis and synthesis). As a matter of fact, the synthesis posed a new and often complicated problem for Descartes.[45]

3.4 An Example of Analysis and Synthesis of an Indeterminate Problem

The indeterminate problem that occupies center stage in the *Géométrie* is the so-called Pappus problem of three or four lines. This problem calls for the construction of a plane curve that satisfies certain conditions. When translated into algebra, as Descartes found, it is reduced to a second-degree algebraic equation in two unknowns. This is the end result of the analytical part of Descartes' problem-solving procedure applied to this problem.

[45] It is highly probable that the constructions were found by the method of indeterminate coefficients.

In the synthetic part Descartes was able to show—a considerable result in his times— that because the equation is second-degree, the locus sought is a conic section.[46]

The solution of the problem of Pappus played an important rhetorical role in the *Géométrie* because Descartes, on the basis of Pappus's account in the seventh book of the *Collectio* that he cited, quite rightly claimed that the ancients could not tackle its generalization to n lines. This boastful statement was challenged by Newton (see chapter 5), who was able to provide a geometrical solution of the four-lines locus "as required by the ancients" and claimed that his solution was simpler and more elegant than Descartes'. The generalization to n lines remained beyond the scope of Newton's geometrical methods, however.

The Pappus problem of three or four lines was generally worded as follows (figure 3.8):

> Having three or four lines given in position, it is required to find the locus of points C from which drawing three or four lines to the three or four lines given in position and making given angles with each one of the given lines the following condition holds: the rectangle of two of the lines so drawn shall bear a given ratio to the square of the third (if there be only three), or to the rectangle of the other two (if there be four).[47]

In short, the Pappus problem of (three) four lines requires, given (three) four lines L_i $(i = 1, 2, 3, 4)$ in the plane, to determine the locus of points C in the plane such that the (positive) oblique "distances" d_i, defined as the lengths of segments drawn from C to L_i at a given angle θ_i, are such that

$$d_1 d_2 = k d_3 d_4, \tag{3.1}$$

where k is a constant.[48]

In the *Géométrie*, Descartes introduced a system of oblique coordinates x and y and, by working with similar triangles, achieved the following result: the locus defined by the problem is a plane curve that is the locus of points satisfying a second-degree algebraic equation in two unknowns. In a greatly streamlined rendering of Descartes' reasoning, one can note that he concluded that the lengths d_i are given by expressions of the form $d_i = \alpha_i x + \beta_i y + \gamma_i$ (where α, β, and γ are constants). Therefore, equation (3.1), which defines the Pappus (three-) four-lines locus, is

[46] Descartes' solution of the Pappus problem has received considerable attention in the literature. See Bos, *Redefining Geometrical Exactness* (2001), pp. 271-83, 313–34.

[47] I paraphrase from *Géométrie*, p. 307 [22]. Descartes could tackle the general problem for any number of lines.

[48] $d_1^2 = k(d_2 d_3)$ if only three lines are given.

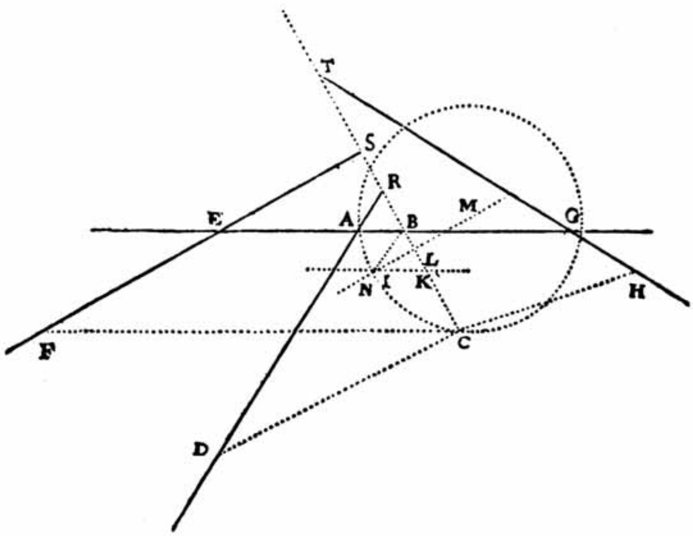


Figure 3.8

Diagram for the Pappus problem. The four straight lines given in position are indicated by solid lines. The oblique distances CD, CF, etc. are dotted lines. The sought locus of points C satisfying equation (3.1) is, in this case, a circle. Source: Descartes, *Géométrie* (1637), p. 327. Courtesy of the Biblioteca Angelo Mai (Bergamo).

second-degree. The algebraic approach easily allowed Descartes to generalize the Pappus problem for any number of lines.[49]

After this analytical stage, Descartes turned to the synthesis, the construction of the locus. His procedure, which occupies center stage in Book 2, can be summarized as follows.[50] Descartes began by constructing a conic section whose position and parameters depend upon the coefficients of the second-degree equation. For the

[49] *Géométrie*, pp. [26-32], [60-3]. Actually, Descartes' algebraic calculation requires a great deal of attention to the signs of the constants and variables that he handled. See Bos's clarification in *Redefining Geometrical Exactness* (2001), pp. 215–6, note 8. Since, in modern terms, the distance of a point $P = (x_0, y_0)$ from a straight line with equation $ax + by + c = 0$ is given by $\pm(ax_0 + by_0 + c)/\sqrt{a^2 + b^2}$, the Pappus problem of four lines should be reduced to an equation of the form $(\alpha_1 x + \beta_1 y + \gamma_1)(\alpha_2 x + \beta_2 y + \gamma_2) = \pm k(\alpha_3 x + \beta_3 y + \gamma_3)(\alpha_4 x + \beta_4 y + \gamma_4)$. The solution, in fact, consists of two curves. Descartes' conventions on signs led him to consider one curve only. The criticisms addressed by Descartes' contemporaries are discussed in Galuzzi and Rovelli, *Nouveauté et Modernité dans les Mathématiques de Descartes* (forthcoming), which I was kindly allowed to see in manuscript.

[50] *Géométrie*, pp. [67–79]. See Bos, *Redefining Geometrical Exactness* (2001), pp. 320–4, for details.

actual construction, Descartes referred to the theory of conics as developed by Apollonius. He then proved that the constructed conic is a solution of the problem by showing that its equation coincided with the equation of the locus.

In this pivotal example, Descartes relied on the classic conic construction consisting in cutting a cone with a plane. He did not use a point-wise construction, a strategy he had previously proposed in Book 1. Examples of constructions of curves by motion appear in the *Géométrie*, too. In some instances, Descartes showed how curves could be generated by the motion of the point of intersections of curves that move according to prescribed rules (the prototype of this method of generation is the quadratrix); in other cases, he referred to motions generated by instruments such as the mesolabum (see figure 3.3) or linked threads (see figure 3.4).

Readers of the *Géométrie* could admire Descartes' success in solving the Pappus problem of three or four lines. Descartes' algebraic approach in principle allowed generalizations to more than four lines, even though Descartes did not work out these generalizations in detail. Moreover, in the *Géométrie* there are no clear indications about how constructions of curves should be performed.

3.5 The Limitations of Descartes' Mathematical Canon

The *Géométrie* made a profound impact, especially after the appearance of its Latin editions. When Newton began his studies in Cambridge, all active mathematicians were referring to it as a seminal text. Descartes had shown how fruitful the application of algebra to geometry could be. This innovative abstract aspect of the *Géométrie* was to change the course of mathematics forever. In his work Descartes also confronted moribund questions stemming from the practices of mathematicians of previous generations, and attempted to answer the many problems raised by Pappus's *Collectio*, which so deeply fascinated early-seventeenth-century mathematicians. Descartes questioned the division into planar, solid, and line-like problems, proposing a much wider class of curves beside the conic sections as possible means for the construction of problems. All geometrical curves were admissible, they were all exact means for constructions. They could be further classified in algebraic terms; their simplicity was defined by the degree of the equation.

There were tensions and open issues in the *Géométrie*, however. Descartes' criteria of demarcation between what is geometrical and exact and what is not, as well as his criteria of simplicity, were problematic. Further, the relation between admissible constructions and polynomial equations was basically unproven. Nowhere in the *Géométrie* was Descartes able to explain which tracing mechanisms were legitimate in generating all and only those curves that are loci of a polynomial equation.

Many doubted that the simplicity of a curve could be reduced to the degree of its equation; simplicity, they claimed, is a geometrical feature that is not obviously captured by the degree of the equation. Criticisms were leveled at Descartes' assumption that all constructions should be carried out in terms of the intersection of a circle and another curve (a straight line, a parabola, a Cartesian parabola, etc.).[51] For example, it was maintained by Guillaume F. A. de L' Hospital that equations of degree 8 should be constructed by two curves of degree 3 rather than by a circle and a curve of degree 4, as Descartes required.[52]

Further, locus problems became more interesting. Most notably, problems whose solutions were curves emerged from all quarters. Curves, more interestingly, were approached as objects of study rather than as means for constructing points by intersection. The determination of tangents to curves and the determination of curvilinear areas were two problems that were becoming ever more prominent in the literature.

At a deeper level, Descartes' insistence that algebra could be applied to geometry when variables and constants occurring in equations are interpreted as finite segments, and the fact that he deployed only finite algebraic equations, seemed to bar the way to the development of techniques that were vital for seventeenth-century mathematicians, especially those interested in the application of mathematics to natural philosophy. As Grosholz wrote:

> Among the most mathematically inventive of Descartes' contemporaries (Cavalieri, Galileo, and Torricelli in Italy, Wallis in England), experimentation with the possibilities of ratios and proportions involving lines and areas, straight and curved lines, and finite and infinitesimal magnitudes (of various dimensions) were already at the forefront of mathematical research that will lead to calculus.[53]

In this passage we find a compact list of aspects of seventeenth-century mathematical research that are not captured by the *Géométrie*. Descartes had prohibited comparisons between straight and curved lines. Further, in the *Géométrie* infinitary techniques and infinitesimals, essential to many seventeenth-century innovative works, were noticeably absent (or perhaps visible only somewhat obliquely in Descartes' method of tangents). Consequently, Descartes had little to say about the rectification of curves; calculation of areas, surfaces, and volumes; or calculations of centers of gravity. One of Descartes' prescriptions that proved to be a serious limitation was his banishment of mechanical curves. As the century progressed, the importance of mechanical curves became more and more evident. They emerged naturally as solutions of what nowadays we identify as problems in integration and the solution of differential equations. They proved useful in mechanics, optics, and astronomy.

In his youth Newton took the *Géométrie* as his starting point. In his maturity he continued to be deeply indebted to this work, which defined his language and his mathematical practice. But he also became its fiercest critic.

---

[51] The Cartesian parabola $axy = (x + a)(x - a)(x - 2a)$ is used in the *Géométrie* to construct fifth- and sixth-degree equations.
[52] Bos, *Redefining Geometrical Exactness* (2001), p. 374.
[53] Grosholz, *Cartesian Method and the Problem of Reduction* (1991), p. 53.

II Against Cartesian Analysis and Synthesis

Just after the creative burst of his *anni mirabiles*, when the tumultuous experience of discovery left very little energy for considerations on exactness and rigor, Newton began posing questions concerning mathematical method, and he did so very seriously. Part II considers a time span from the late 1660s to the early 1680s. This period was a turning point in Newton's intellectual career. His new status as Lucasian Professor (1669) required an approach different from the unsystematic forays into new territory permitted the young protégé of Barrow. The encounter in 1673 with Huygens's *Horologium Oscillatorium* was probably another factor that induced Newton to think about the importance and beauty of geometry when compared with the Cartesian symbolic analysis. Huygens showed how natural philosophy could be mathematized in terms consonant with the standards of certainty that Newton attributed to the methods of the ancient geometers. Thus, Part II begins to probe the conflict that Newton faced when he compared the mathematical practices of the moderns, in which he excelled, with the role he attributed to geometry in his philosophical agenda. Newton never ceased to praise Huygens and to indicate the *Horologium* as a model in mathematical method. More generally, he began elaborating a deep distaste for all things Cartesian, and he did so for reasons surpassing mathematical enquiry. In this period he might also have begun to mature his veneration for the ancients that pervades his alchemical and theological work. It seems that almost everything in Newton's mind was pointing against Cartesianism.

Chapter 4 describes Newton's deep criticisms against the Cartesian approach to determinate problems. In particular, in *Lucasian Lectures on Algebra*, which Newton completed in 1684, he rejected the Cartesian canon for the construction of equations via the intersection of curves. *Contra* Descartes, algebraic criteria have no place in composition, Newton argued. Therefore, Descartes' demarcation between legitimate and illegitimate curves could not be accepted because it was ultimately based, according to Newton, on the equations associated with them.

Chapter 5 devotes attention to what Newton had to say against Descartes' approach to indeterminate problems. It considers "Errores Cartesij Geometriae" and "Solutio Problematis Veterum de Loco Solido." In this context Newton came to believe that the ancients possessed a method of discovery based on projective geometry that allowed them to deal with locus problems in a way more satisfactory than Descartes'.

In the 1670s, Newton also worked on cubic curves. His enumeration of third-degree curves is an exercise in Cartesian algebra; indeed, in his work on cubics Newton deployed tools, such as infinite series, which he perceived as more advanced and modern than those contemplated in the *Géométrie*. Chapter 6 describes how Newton dealt with this tension between his methodological views and his mathematical practice. In analyzing *Enumeratio Linearum Tertii Ordinis*, a work written in the mid-1690s in which Newton systematized his researches on cubics carried out in the 1670s, I identify a typical characteristic of his policy of publication, namely, the reluctance to reveal the algebraic analysis that allowed some of his greatest advancements in mathematics.

4 Against Descartes on Determinate Problems

> Composition is perfect in itself and shrinks from any admixture with analytical speculations.
>
> —Isaac Newton, early 1680s

> [The Ancients] distinguished resolution from solution from one another as dual converses ... regarding a problem as resolved when a geometer had in his own view completed its analysis, and as solved once he had without analysis learned how to compose it.
>
> —Isaac Newton, 1690s

4.1 *Lucasian Lectures on Algebra*

Sometime between the autumn of 1683 and early winter of 1684, Newton, according to the statutes of the Lucasian Chair, deposited his *Lucasian Lectures on Algebra*.[1] The lectures bear dates from 1673 to 1683, but these were added in retrospect, and it is highly unlikely that they were ever delivered to Cambridge students. From several points of view, and notwithstanding Newton's professed anti-Cartesianism, these lectures can be described as a fulfillment of Descartes' program because algebra is here extensively presented as the tool to be used in the analysis (or resolution) of problems. Indeed, the lectures were printed in 1707 with a revealing title: *Arithmetica Universalis: Sive de Compositione et Resolutione Arithmetica Liber*.[2] The title page leaves little doubt as to the content of this widely read book; it

Epigraph sources: (1) MP, 5, p. 477. "Compositio in se perfecta est et a mixtura speculationum Analyticarum abhorret." MP, 5, p. 476. (2) MP, 7, p. 251. "[Veteres] Resolutionem et Solutionem ut contraria duo ad invicem distinguebant ... existimantes Problema resolutum esse quando Geometra apud se absolverat Analysin, solutum quando sine Analysi componere didicerat." MP, 7, p. 250.

[1] Cambridge University Library, Dd.9.68. The manuscript (probably written in 1683–4) bears no title: it is here referred to as *Lucasian Lectures on Algebra*. The edition of this 251-folios long manuscript is in MP, 5, pp. 54–491.

[2] The title was chosen by the editor, William Whiston, but was retained by Newton in his revised edition (1722). The title "Arithmeticae Universalis Liber Primus" occurs in a partial revised version, Add. 3993, f. 1 (see MP, 5, p. 538). Note the inversion in the title between compositio and resolutio, one would expect resolutio first. It seems that compositio (which Newton conceived of as being purely geometrical) is given pride of place. Further, the adjective arithmetica is meant to be associated with resolutio, not compositio.

was meant to be a work devoted both to algebraic analysis (*arithmetica resolutio*), which translates problems into equations, and to synthesis (*compositio*), by means of which problems are geometrically constructed.[3]

Newton began working on Cartesian algebra quite early. His interest in the field is shown by the extraordinary array of results that he soon achieved.[4] Most notably, Newton extended Descartes' rule of signs by developing a rule for enumerating the imaginary roots that was proven by James J. Sylvester in the nineteenth century.[5]

Newton's next work on algebra contained the observations on Gerard Kinckhuysen's textbook that he prepared in 1670 for Collins.[6] Kinckhuysen's algebra—Mercator's Latin translation of the Dutch elementary treatise—with Newton's commentary was never published, but Newton nevertheless employed it for his Lucasian lectures.[7]

In his lectures Newton extended his observations on Kinckhuysen by adding two new sections: "How Geometrical Questions are to Be Reduced to an Equation," and "The Linear Construction of Equations."[8] These two sections reveal Newton's concern with the status and role of algebra, a matter that occupied him in the 1670s. For this reason, I discuss them in some detail.

Stedall, in her useful commentary on the reception of *Arithmetica Universalis*, summarizes the theoretical results pertinent to the theory of equations as follows:

1. A method for finding factors of polynomials, under the heading "De Inventio Divisorum" (42–51).
2. Rules for finding roots of quantities of the form $A + \sqrt{B}$ (58–61).
3. Rules for transforming two or more equations into one. Almost all of this short section is devoted to the case of two equations in two unknowns (69–76).
4. Descartes' "rule of signs" already gave an upper bound for the number of positive roots of an equation by examining changes of sign. Newton added a new rule for determining the number of "impossible" or imaginary roots (242–245).

---

[3] Note that Newton employed the term universal arithmetic for algebra because it is concerned with the doctrine of operations, applied not to numbers but to general symbols.

[4] In May 1665, Newton wrote about the geometrical construction of equations; in 1666 he wrote about the theory of equations. MP, 1, pp. 489–502, 506–39.

[5] Pycior, *Symbols, Impossible Numbers, and Geometric Entanglements* (1997), pp. 172–4; Bartolazzi and Franci, "Un Frammento di Storia dell'Algebra" (1990); Parshall, *James Joseph Sylvester* (2006).

[6] See Whiteside's commentary in MP, 2, pp. 277–94 and Newton's observations (Add.3959.1, ff. 2r–21r) in MP, 2, pp. 364–445.

[7] Mercator's translation (interleaved in a copy of Kinckhuysen's book held at the Bodleian Library, Oxford (Savile G.20 (4))) is transcribed in MP, 2, pp. 295–364.

[8] "Quomodo Quaestiones Geometricae ad Aequationem Redigantur." MP, 5, pp. 158–337. "Aequationum constructio linearis." MP, 5, pp. 420–91, derived from an earlier manuscript edited in MP, 2, pp. 450–517 with the editor's title "Problems for constructing equations."

5. The rules for expressing coefficients of equations as symmetric functions of the roots were well known by now. Newton used these functions to find formulae for sums of powers of the roots (251–252).
6. Rules for finding the "limits" or bounds between which the roots of an equation must lie (252–257).
7. Techniques for solving equations whose factors might contain surd quantities (257–272).[9]

This is an impressive list. However, *Arithmetica Universalis* contains much more.

When, in 1707, the lectures appeared under the title of *Arithmetica Universalis*, readers were somewhat puzzled. Indeed, since then, historians of Newtoniana have had a hard time trying to pigeonhole the work as being either algebraic or geometric. Pycior, one of the best historians of the subject, has spoken of a "mixed mathematical legacy" left by Newton to his disciples.[10] From one point of view *Arithmetica Universalis* can be seen as a fulfillment of the program outlined by Descartes in the *Géométrie* because it teaches how problems, especially geometrical problems (but also arithmetical and mechanical ones), can be translated into the language of algebra, which is here seen as the tool for problematic analysis; on the other hand, *Arithmetica Universalis* contains two criticisms directed at Descartes.

First, in a long section devoted to the reduction of geometrical problems to an equation, Newton distanced himself from Descartes. He maintained that, at least in some cases, Apollonian geometry is to be preferred to Cartesian algebra in the analysis of indeterminate problems.[11]

Second, in the last section of the work (edited by Whiston in 1707 as an Appendix), a section devoted to the construction of equations, Newton argued that the demarcation between acceptable and nonacceptable means of construction of determinate problems, as well as the characterization of the relative simplicity of such means proposed by Descartes, were far too dependent upon algebraic criteria (namely, the existence of a polynomial equation defining such means, and the degree of the equation).

This chapter examines Newton's critical approach to the Cartesian construction of equations. Newton's approach to indeterminate problems is the subject of chapter 5.

---

[9] Stedall, "Newton's Algebra" (forthcoming). Page numbers refer to Newton, *Arithmetica Universalis* (1707).

[10] Pycior, *Symbols, Impossible Numbers, and Geometric Entanglements* (1997), p. 167.

[11] MP, 5:, p. 314 *passim*; see also chapter 5. Newton also explored the algebraic way with great flexibility and creativity. Indeed, *Arithmetica Universalis* is a rich repertoire of alternative algebraic approaches to geometrical and mechanical problems.

4.2 Demarcation and Simplicity in the Construction of Equations

Lucasian Lectures on Algebra ends with a section (the final Appendix of *Arithmetica Universalis*) devoted to the linear construction of equations, which abounds with statements in favor of geometry and is directed against the moderns, who have lost the "elegance of geometry":

> [F]or anyone who examines the constructions of problems by the straight line and circle devised by the first geometers will readily perceive that geometry was contrived as a means of escaping the tediousness of calculation by the ready drawing of lines. Consequently these two sciences [arithmetical computation and geometry] ought not to be confused. The Ancients so assiduously distinguished them one from the other that they never introduced arithmetical terms into geometry; while recent people, by confusing both, have lost the simplicity in which all elegance of geometry consists.[12]

Such statements have often puzzled commentators because they occur in a work devoted to algebra that illustrates the advantage of algebraic analysis in a long section on the reduction of geometrical problems to equations. Why was Newton turning his back on universal arithmetic now by arguing that algebra and geometry should be kept separate? In order to understand Newton's seemingly paradoxical position, it is useful to briefly review the stages in the analytical and synthetic processes for the solution of determinate problems endorsed by the Cartesian school.

Descartes' canon was divided, in accordance with Pappusian tradition as well as with early-seventeenth-century mathematical practice, into analytical and synthetic stages. The analytical stage was to be carried on through algebra. The end result of the analysis of determinate problems (such as angle trisection) was an algebraic equation in one unknown. One of the elements in the synthetic stage of Descartes' canon to which Newton devoted great attention, was the construction of equations. These constructions had to be carried on in terms of intersections of admissible curves, and among the admissible curves one had to choose the simplest. Ultimately, Descartes was forced to employ algebraic criteria of demarcation and simplicity. The demarcation between admissible and inadmissible curves as means of construction was that between geometrical (algebraic) and mechanical (transcendental) curves. Geometrical curves coincided with the loci of polynomial equations; the degree of the equation allowed the ranking of geometrical curves in terms of their simplicity. As Bos has remarked, Descartes, in discussing means of construction, was forced to

[12] MP, 5, p. 429. "Nam qui constructiones Problematum per rectam et circulum a primis Geometris adinventas considerabit facile sentiet Geometriam excogitatam esse ut expedito linearum ductu effugeremus computandi taedium. Proinde hae duae scientiae confundi non debent. Veteres tam sedulo distinguebant eas ab invicem ut in Geometriam terminos Arithmeticos nunquam introduxerint. Et recentes utramque confundendo amiserunt simplicitatem in qua Geometriae elegantia omnis consistit." MP, 5, p. 428. Compare with Fermat's criticism of Wallis (§7.2).

rely on algebra because his attempts to define curve-tracing methods that give rise to geometrical curves led to a result that in the eyes of his critics, including Newton, appeared to be a failure, concealed with some difficulty in the complex structure of the *Géométrie*.[13]

As far as demarcation is concerned, in *Lucasian Lectures on Algebra*, Newton maintained that it would be wrong to think that a curve can be accepted or rejected on the basis of its defining equation:

> [I]t is not the equation but its description which produces a geometrical curve. A circle is a geometrical line not because it is expressible by means of an equation but because its description (as such) is postulated.[14]

Moreover, Newton claimed, Descartes' classification of geometrical curves according to the degree of the equation is not relevant for the geometrician, who should choose curves on the basis of the simplicity of their description. Newton, for instance, observed that the equation of a parabola is simpler than the equation of a circle. However, it is the circle that proves simpler and is to be preferred in the construction of problems:

> It is not the simplicity of its equation, but the ease of its description, which primarily indicates that a line is to be admitted into the construction of problems. ... On the simplicity, indeed, of a construction the algebraic representation has no bearing. Here the descriptions of curves alone come into the reckoning.[15]

Newton observed that, from his point of view, the conchoid, a fourth-degree curve, is quite simple. Aside from any considerations about its equation, its mechanical description, he claimed, is one of the simplest and most elegant; only that of the circle is simpler. Descartes' algebraic criterion of simplicity was thus regarded as alien to the constructive, synthetic stage of problem solving.

The weakness of Newton's position is that the concepts of simplicity of tracing or of elegance, to which he continually referred, are qualitative and subjective. No compelling reason is given in support of Newton's evaluations of the simplicity of his preferred constructions; his criteria are largely aesthetic. It is nevertheless crucial to take them into account in order to understand Newton's methodology. In what follows, therefore, the themes of demarcation and simplicity presented by Newton in the final section of *Lucasian Lectures on Algebra* are explored in more detail.

---

[13] See Bos, *Redefining Geometrical Exactness* (2001), p. 402.

[14] MP, 5, p. 424. "At aequatio non est sed descriptio quae curvam Geometricam efficit. Circulus linea Geometrica est non quod per aequationem exprimi potest sed quod descriptio ejus postulatur." MP, 5, p. 425.

[15] MP, 5, pp. 425, 427. "Aequationis simplicitas non est sed descriptionis facilitas quae lineam ad constructiones Problematum prius admittendam esse indicat. ... Ad simplicitatem vero constructionis expressiones Algebraicae nil conferunt. Solae descriptiones linearum hic in censum veniunt." MP, 5, p. 424.

The title of the final section of Newton's work, "The Linear Construction of Equations," is programmatically anti-Cartesian.[16] Newton here dealt exclusively with a very particular example: the construction of third-degree equations that Descartes performed via intersection of circle and parabola (§3.3). Newton proposed instead to use a curve of degree higher than the conics as a means of construction: the conchoid (a fourth-degree curve). Here is where the term *linear* comes from. According to Pappus's terminology, plane constructions are carried out by means of circles and straight lines; solid ones by means of circles, straight lines, and conic sections; and linear ones by means of more complex curves.[17] Newton defended the legitimacy of transgressing Descartes' criteria of simplicity by admitting the construction of third-degree equations carried out through curves of higher degree than the conic sections. These constructions are none other than the classic neusis (§4.3). Basing his position on the authority of the ancients, notably Archimedes, and citing Pappus, Newton wrote:

> So, after the Ancients had achieved the accomplishment of these problems [the classic problems of angle trisection and cube duplication] by compounding solid loci, feeling that constructions of this sort are, because of the difficulty of describing conics, of no practical use, they looked for easier constructions by means of the conchoid and the cissoid, the extending of threads and any kind of mechanical application of figures: as we learn from Pappus, mechanical usefulness was preferred to useless geometrical speculation. Thus the mighty Archimedes ignored the trisection of the angle by means of conics expounded by his predecessors in geometry, and in his *Lemmas* taught how to cut an angle into three by the method we just now exhibited. If the Ancients preferred to construct problems by means of figures not at that period received into geometry, how much greater should our preference now be for those figures when by most they are received into geometry on an equal footing with conics themselves?[18]

The "method just now exhibited," taken from Archimedes' *Lemmata or Liber Assumptorum*, which Newton probably knew from Barrow's edition,[19] is a construction of the angle trisection problem obtained via the tracing of a conchoid (§4.3).

Newton even defended the use of mechanical curves in the construction of problems. It is in this context that he defended the use of the trochoid (cycloid), a

[16] As Whiteside points out, Newton deleted this title in his private copy of *Arithmetica Universalis*. He also deleted most of the Appendix, with the exception of two neusis constructions. These changes, however, were not implemented in the 1722 Latin edition that Newton supervised. The title of the published *Arithmetica Universalis* was chosen by Whiston, perhaps after consulting Newton. See Whiteside's comments in MP, 5, pp. 14, 421 (615n).

[17] Pappus, *Collectionis Quae Supersunt* (1876–78), 3, pp. 38–9. The reader should carefully distinguish between the meaning of the term *linear* as used by Newton and its meaning in twenty-first-century mathematics.

[18] MP, 5, pp. 469–71.

[19] Barrow, *Archimedis Opera* (1675), pp. 265–76.

mechanical curve excluded from geometry by Descartes' criteria, in the construction of the general angle section:

> Were the cycloid to be accepted into geometry, it would be allowable by its aid to cut up an angle in a given ratio. Could you then, if someone were to use this line to divide an angle in an integral ratio, see anything reprehensible in this and contend that this line is not defined by an [algebraic] equation, but that lines defined by equations need to be employed? ... a curve which is exceedingly well known and very easily described through the motion of a wheel or a circle. How absurd this is, any one may see. Either, then, the cycloid is not to be admitted into geometry; or in the construction of problems it is to be preferred to all curves having a more difficult description.[20]

Newton's disagreement with Descartes' criteria of demarcation could not be any stronger. According to Newton, mechanical curves are quite acceptable in geometrical constructions provided that their mechanical generation is simple. This is the case with the cycloid, which can be used as a means of geometrical construction (in this case, in constructing a section of a circular arc in any given ratio), even when it has no defining polynomial algebraic equation in Cartesian coordinates.

The existence and nature, or degree, of defining equations is something totally alien to geometry. Descartes dealt with the problem of angle trisection by reducing the problem to a third-degree equation and constructing it via intersection of circle and parabola (§3.3.3). Newton observed that if mechanical curves such as the cycloid, the quadratrix, or the Archimedean spiral are accepted as means of construction, then it is possible to divide an angle into any ratio.[21]

Before continuing the discussion on Newton's methodology for the construction of algebraic equations, I briefly consider which constructions Newton prescribed. In his treatment of the construction of third-degree equations, Newton considered two approaches: the first implies the use of the conchoid (§4.3), the second the use of the circle and the ellipse (§4.4). In both cases the curves are conceived of as being traced by motion. Newton stressed that in dealing with constructions he conceived of curves as being mechanically generated by tracing devices and paid no attention to their algebraic definition. In this context he also provided a mechanical construction of the cissoid, a curve employed by the ancients to find two mean proportionals (see figure 4.5). It is interesting to see how Newton defended the use of these curves as means of construction as well as the reason he gave for preferring mechanical constructions in which curves are generated by motion.

---

[20] MP, 5, p. 427.

[21] As Pappus had shown in the case of the spiral and the quadratrix in Propositions 45 and 46, Book 4. Pappus, *Collectionis Quae Supersunt* (1876–78), 1, pp. 284.21–288.14.

4.3 Neusis Constructions

Descartes admitted all geometrical (algebraic) curves as means of construction. Yet geometrical curves, even third-degree ones, can be quite complicated to trace (see chapter 6); not all third-degree curves possess the aesthetic simplicity of tracing that is required in geometrical constructions. So Newton proposed to base all means of construction on curves generated by continuous motions that are easy to perform. One of the best choices is the conchoid, according to Newton the simplest curve after the circle. Admitting the conchoid as a means of construction allowed Newton to construct third-degree equations. He first showed in a Lemmatical Problem that if the conchoid is given, all neusis constructions are possible.[22] Here Newton referred to the fourth book of Pappus's *Collectio*, but he probably also had in mind Viète's *Supplementum Geometriae*.[23]

Pappus introduced the conchoid as a means for the construction of two problems: the finding of two mean proportionals and the trisection of the angle.[24] The conchoid is defined as follows (figure 4.1):

> **Definition of the conchoid**: Given a line l, a point O at distance d from l, and a segment a of length k. Let A be an arbitrary point on l, and P, P' the points on the line OA at distance k from A. The locus of all these points P, P' is a conchoid.[25]

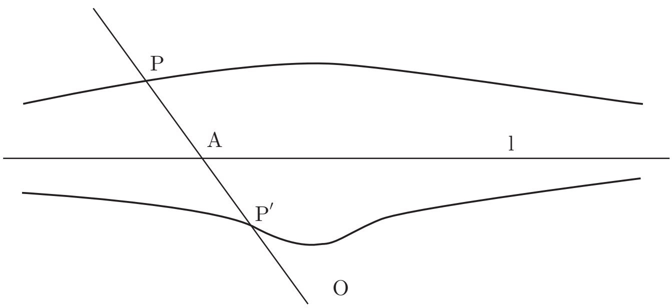


Figure 4.1
Conchoid for $k < d$, where k is the length of $AP = AP'$ and d is the distance of O from l. The equation of the conchoid, for an orthogonal coordinate system where the line l is the x-axis, and the y-axis passes through O, is $x^2y^2 = (d + y)^2(k^2 - y^2)$. Source: Brieskorn and Knörrer, *Plane Algebraic Curves* (1986), p. 14. ©1981 Birkhäuser Boston Inc. With kind permission of Springer Science and Business Media.

---

[22] MP, 5, pp. 429–30.
[23] In Viète, *Opera Mathematica in Unum Volumen Congesta ac Recognita* (1646), pp. 240–57.
[24] Bos, *Redefining Geometrical Exactness* (2001), pp. 33, 55.
[25] Brieskorn and Knörrer, *Plane Algebraic Curves* (1986), p. 13.

The conchoid could be used, as Pappus showed, to solve the neusis problem (figure 4.2):

> **Definition of neusis problem**: Given two straight lines L and M, a point O and a segment a of length k; it is required to find a line through O, intersecting L and M in A and B such that $AB = a$.[26]

Neusis means verging: the given segment a is placed between the two given lines so that it verges toward the given pole.

Mechanical constructions of the conchoid were known. The conchoid is a curve that can be traced by the continuous movement of a linked ruler like the one shown in figure 4.3.

Interest in neusis construction was stirred by the publication of Pappus's *Collectio* in 1588, which proved that both the problem of finding two mean proportionals (equivalent to the duplication of the cube) and that of angle trisection could be constructed if one admits a neusis construction. Most notably, among early-modern mathematicians, Viète assumed neusis as a new postulate: he argued that constructions carried on by means of neusis should be postulated alongside those carried on by ruler and compass. Viète also showed that any geometrical problem leading to a third- or fourth-degree algebraic equation could be reduced either to finding two mean proportionals between two given lines or to trisecting a given angle and therefore could be constructed by neusis.

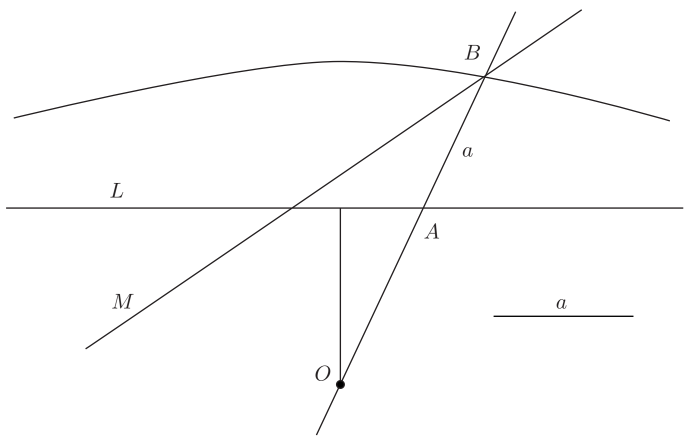


Figure 4.2

Neusis by means of the conchoid. Source: Bos, *Redefining Geometrical Exactness* (2001), p. 32. ©2001 Springer-Verlag New York Inc. With kind permission of Springer Science and Business Media.

[26] Bos, *Redefining Geometrical Exactness* (2001), p. 31.

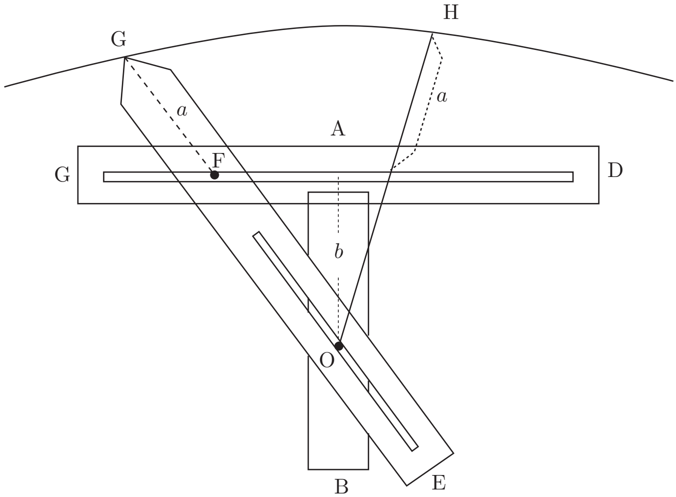


Figure 4.3
Organic construction of the conchoid. A horizontal ruler is fixed in position; it has a slot parallel to its length and a vertical ruler mounted at right angles with it. A peg at O is fixed on the vertical ruler. A peg at F is fixed on the movable ruler. Peg F slides in the horizontal slot while peg O slides in the slot cut into the movable ruler. Point G organically describes the conchoid. Source: Bos, *Redefining Geometrical Exactness* (2001), p. 31. ©2001 Springer-Verlag New York Inc. With kind permission of Springer Science and Business Media.

It is this tradition that Newton referred to in his work. Whereas Descartes had preferred construction of third- and fourth-degree algebraic equations by means of conic sections (parabola and circle), curves of second degree (§3.3), Newton proposed to construct them by means of the conchoid, a fourth-degree curve. Reading Pappus through the eyes of Viète, Newton took a critical position toward Descartes: he argued that the tracing of a higher-degree curve can be admitted as a postulate, which enables constructions that are more elegant and simple than those of Descartes.

Descartes reduced the problem of angle trisection (§3.3.2) to an equation of the form:

$$x^3 = qx + r. \tag{4.1}$$

The construction of a third-degree equation is achieved by Newton through neusis construction (figure 4.4):

> *Let there be proposed the cubic equation $x^3 + qx + r = 0$, whose second term is lacking*, but [the coefficient of] the third is denoted under its proper sign by $+q$ and the fourth by $+r$.

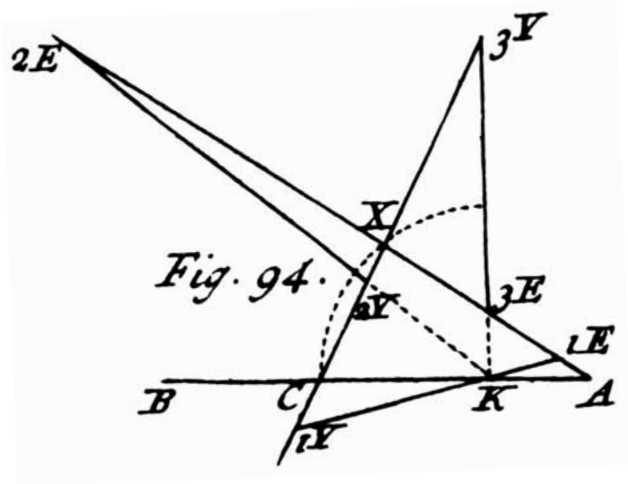


Figure 4.4
Construction of $x^3 + qx - r = 0$ by means of the conchoid in *Arithmetica Universalis*. The construction implies that $KA = n$, $KB = q/n$, $CX = r/n^2$. Newton distinguished several cases according to the number and sign of the real roots, which correspond to the lengths of segments XY. This figure illustrates the case in which there are three real roots. Accordingly, three insertions of EY verging towards K are possible. They are labeled by subscripts. Source: Newton, *Universal Arithmetick* (1720), Tab. VII. Courtesy of the Biblioteca Angelo Mai (Bergamo).

> Draw any [straight line] KA you please and call n [the length of segment KA]. In KA extended in either direction take $KB = q/n$: the same way as KA if $+q$ be had, but otherwise the opposite way. Bisect BA in C and with center K, radius KC construct the circle CX and in it inscribe the straight line CX equal to r/n^2, producing it each way. Next, join AX and extend it each way.[27] Finally, between the lines CX and AX inscribe [by neusis] EY of the same length as CA such that, when produced, it shall pass through the point K, and XY will be a root of the equation. Of these roots the ones falling on the side of X towards C will be positive and those falling on the opposite side negative if $+r$ be had, but the converse if $-r$ be supposed.[28]

This construction was to be followed by a geometrical demonstration; namely, a geometrical proof that the construction answers the problem. No hint is given, however, about how the geometrical construction was found.[29]

---

[27] Until this stage all constructions can be performed by ruler and compass.

[28] MP, 5, pp. 433, 435. I have interpolated Whiteside's translation with some explanations in square brackets.

[29] In his study of this aspect of Newton's work, Bos has identified the way in which the construction

Newton's demonstration is based on three geometrical lemmas.[30] From the first and third the following relation ensues:

$$KA^2 \times CX - KA \times KB \times XY = XY^3, \tag{4.2}$$

which is immediately translated into the language of algebra as

$$n^2 \frac{r}{n^2} - n\frac{q}{n}x = r - qx = x^3. \tag{4.3}$$

4.4 Conic Constructions

The other method of construction of third-degree equations proposed by Newton is by the intersection of circle and ellipse. One might think that Newton ought to have considered constructions by circle and ellipse as preferable to those performed by conchoid. In fact, the conchoid, one could argue, is less exact than the circle and the ellipse, since its definition is mechanical rather than geometrical. The circle is given by postulate, and the ellipse can be described in purely geometrical terms as the section of a cone. Mechanical descriptions of curves (organic descriptions) were often regarded with some suspicion. But Newton did not reason like that; for him, all curves, even the circle, were primarily given by mechanical descriptions. In a treatise on geometry that Newton composed in the 1690s, he wrote:

> [I]n definitions [of curves] it is allowable to posit the reason for a mechanical genesis, in that the species of magnitude is best understood from the reason for its genesis.[31]

Curves that are posited mechanically are better understood because one knows the "reason for their genesis." Newton, for his own idiosyncratic purposes, is here deploying terminology familiar to readers of the fourth book of Pappus's *Collectio*, where the properties (*symptomata*) of mechanical lines such as the Archimedean spiral, the conchoid of Nicomedes, the quadratrix of Dinostratus and Nicomedes,

---

was found. His very plausible guess is that Newton used the algebraic method of undetermined coefficients. Bos, "Arguments on Motivation in the Rise and Decline of a Mathematical Theory" (1984), p. 361.

[30] The first two lemmas are (i) $XY/KA = CX/KE$, (ii) $XY/KA = CY/(KA + KE)$. In fact, the second lemma is an easy consequence of the first. From the second lemma and *Elements*, II, 12, a third lemma follows: (iii) $XY/KA = (KE - KB)/XY$.

[31] MP, 7, p. 291. "In definitionibus ponere licet rationem geneseos Mechanicae, eo quod species magnitudinum ex ratione geneseos optime intelligitur." MP, 7, p. 290.

and a spiral on the surface of a sphere were proven by means of the defining mechanical genesis of the line. Newton thus provided a mechanical description of the ellipse by the trammel and constructed third-degree equations by intersections of circle and ellipse.

Descartes had proposed a canon for the construction of third-degree equations in terms of intersections between a circle and a parabola. Such intersections could not be performed via algebraic means in a general way. The intersection could exactly be determined only by means of geometrical constructions. Descartes, however, did not introduce the parabola and the circle via a tracing mechanism. In his discussion of the geometrical construction of conics he chose rather to consider the conic sections as given in terms of the sections of a plane with a cone.[32] Newton disagreed on this point. In Newton's terminology, curves could be described either geometrically (via a geometrical relation, e.g., that between a plane and a cone) or organically (via a mechanical description). In the construction of problems, according to Newton, one is bound to employ mechanical definitions of curves; these alone ensure that the point of intersection between the curves is really given, something ensured by the continuity of tracing. Thus, in order to construct third-degree equations by means of conic sections rather than a conchoid, one needs (i) a mechanical description of conics, and (ii) a criterion of simplicity based on this mechanical description. Curves defined geometrically (in this case as sections of the cone) are, in Newton's words, "useless speculation"; curves are useful for constructions when a mechanical, or organic generation is known.

Naturally, Newton's construction by ellipse and circle is not that different from Descartes' construction by parabola and circle. Indeed, Newton showed how Descartes' construction could be derived from his own. He also emphasized, however, that his construction was carried on in terms of conics, which are generated mechanically, whereas Descartes relied on a geometrical definition of conics as sections of a cone.

But why choose the ellipse rather than the parabola? This question points to Newton's last attack against the Cartesian canon:

> So far I have expounded the construction of a cubic equation by an ellipse only: but by its very nature the rule is more general, extending to all conics indifferently. ... Thus, constructions by a parabola [namely, the ones proposed by Descartes] are, if you regard their analytical simplicity, simplest of all; those by the hyperbola hold next place; and the last place is retained by those accomplished by an ellipse. But if regard be had for the simplicity of their manual procedure in describing figures, that order must be inverted.[33]

[32] Bos, *Redefining Geometrical Exactness* (2001), p. 320.
[33] MP, 5, pp. 485–7.

4.5 Against Cartesian Synthesis

The message that Newton wished to deliver in the final section of *Lucasian Lectures on Algebra*, devoted to the linear construction of equations, is that in geometrical constructions algebraic criteria are misleading. Descartes had admitted all geometric curves as means of construction. Most of these, however, were hopelessly complex. On the other hand, some mechanical curves, such as the cycloid, were simple means of construction. Further, the hierarchy of simplicity given in terms of the degree of algebraic equations is foreign to geometry. Circle and ellipse are of the same degree, but the former is simpler. The conchoid is a curve of fourth degree, but admitting it as a postulate is equivalent to admitting neusis constructions, which, in the construction of third-degree equations, prove to be better than conic sections. When practicing geometry, Newton insisted, curves (even the cissoid, which was traditionally constructed point-wise) must be seen as being traced by motion (figure 4.5).

Newton's argument has several weak points; his criteria for geometrical construction are vague and essentially based on aesthetic preferences, whereas Descartes was able to provide a canon that is defined in clear mathematical terms. Why, for instance, should one consider a neusis construction any more elegant than one achieved via intersection of circle and parabola? Further, Newton's idea that one should accept such mechanical curves as the spiral and the cycloid in the construction of problems such as angle sections trivializes Descartes' effort to distinguish between acceptable and unacceptable means of construction in such a way that in constructing a problem one should never use means as difficult to construct as the problem itself. For Descartes admitting the cycloid or the spiral for angle section would have meant using a means of construction that presupposes the construction of the problem at hand.

Notwithstanding such weaknesses, Newton's fiery invectives against the Cartesian method, which abound in the final section of *Lucasian Lectures on Algebra*, are not paradoxical, as is often claimed. Newton made it clear that in the section on the linear construction of equations he was talking about the synthetic, constructive phase of the problem-solving process; he was "treating of composition." The analytical stage, discussed in the section devoted to the reduction of geometrical questions to equations (see chapter 5), can be carried on in algebraic terms. Indeed, in the *Lectures*, algebra is proposed as one of the admissible analytical tools. In the synthetic, constructive stage, however, algebra must not play any role:

> Present-day geometers indulge too much in speculation from equations. The simplicity of these is a consideration belonging to analysis: we are here [in the final section of *Lucasian Lectures on Algebra*] occupied with composition, and laws are not to be laid down for composition from an analytical standpoint. Analysis guides us to the composition, but true composition is not achieved before it is freed from

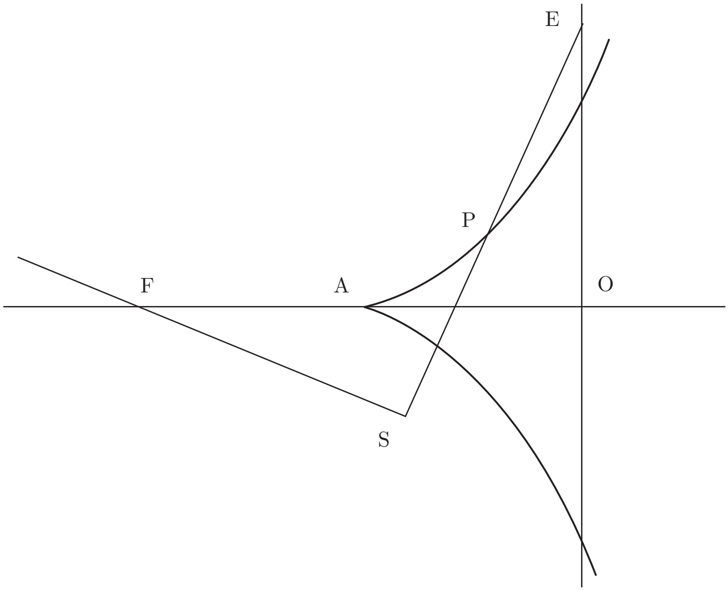


Figure 4.5
Newton highly valued organic descriptions of curves. He invented a mechanical description of the cissoid. The original in MP, 5, pp. 465–7 (see also MP, 7, pp. 390–1) was emended in the printed *Arithmetica Universalis*). A right angle with arm of fixed length $2r$ is moved in the plane in such a way that its end point E moves on a line and the other arm always goes through a fixed point F at distance $2r$ from the line. The midpoint P of the arm SE then describes the cissoid. See §6.1 and §8.4.5 for the properties and defining third-degree equation of the cissoid. Source: Brieskorn and Knörrer, *Plane Algebraic Curves* (1986), p. 12. ©1981 Birkhäuser Boston Inc. With kind permission of Springer Science and Business Media.

> analysis. Let even the slightest trace of analysis be present in the composition and you will not yet have attained true composition. Composition is perfect in itself and shrinks from an admixture with analytical speculations.[34]

Again, this position does not exclude the use of algebra in the stage of analysis; however it rules out algebraic criteria of demarcation and simplicity from the synthesis. Newton formulated a clear partition between discovery of the equation, its

[34] "Aequationum speculationi nimium indulgent hodierni Geometrae. Harum simplicitas est considerationis Analyticae. Nos in compositione versamur et Compositioni leges dandae non sunt ex Analysi. Manuducit Analysis ad Compositionem: sed Compositio non prius vera confit quam liberatur ab omni Analysi. Insit Compositioni vel minimum Analyseos et Compositionem veram nondum assecutus es. Compositio in se perfecta est et a mixtura speculationum Analyticarum abhorret." MP, 4, p. 477.

construction, and a complete demonstration in a manuscript dating from the early 1690s:

> [I]f a question be answered by the construction of some equation, that question is resolved by the discovery of the equation and composed by its construction, but it is not solved before the construction's enunciation and its complete demonstration is, [a]from beginning to end excluding analysis perfectly[a] with the equation now neglected, composed. Hence it is that resolution so rarely occurs in the ancients' writings outside Pappus's Collection.[35]

4.6 Against Cartesian Analysis

Newton not only criticized the Cartesian synthesis of determinate problems, but also expressed some reservations with regard to Cartesian analysis. He positioned himself against the Cartesian canon of geometrical problem construction via the intersection of geometrical curves, and he also disliked algebra as an analytical means of discovery. When Newton turned to indeterminate (or locus) problems, he embarked on a sustained campaign against Cartesian algebraic methods. As for determinate problems, his position was more nuanced. While recognizing the usefulness of the Cartesian algebraic approach, even in the case of determinate problems he showed a predilection for geometrical analysis.

In Descartes' *Géométrie*, Newton encountered the thesis that algebra is the tool to be employed in the analysis of geometric problems. The largest part of *Lucasian Lectures on Algebra* is devoted to developing this kind of Cartesian problematic analysis. However, already while preparing his notes on Kinckhuysen, Newton had begun to compare geometrical and algebraic analyses. Newton's annotations on this topic reveal that he progressively deepened his dislike for the latter. The main criticism that he leveled is that algebraic analysis does not reveal how the geometrical synthesis can be performed. After the geometrical analysis of a problem it is possible to reach a construction by reverting the steps of the analysis, but after an algebraic analysis one is left with an additional and artificial problem: Descartes' problem of the construction of the equation.[36] Newton concluded that such constructions were largely a Cartesian contrivance with no roots in the ancient geometrical tradition. Further, these constructions were un-

35 [aa] canceled. Newton, MP, 7, p. 307. "si quaestioni per constructionem aequationis alicujus respondeatur, quaestio illa resolvitur per inventionem aequationis, componitur per constructionem ejusdem, sed non prius solvitur quam constructionis enunciatio ac demonstratio tota [a]a principio ad finem exclusa omni analysi perfecte[a] componitur, aequatione neglecta. Hinc est quod resolutio in veterum scriptis extra Pappi collectanea tam raro occurrat." MP, 7, p. 306.

36 Brigaglia, "La Riscoperta dell'Analisi e i Problemi Apolloniani" (1995).

naturally complicated and less elegant than those of the ancients. In the 1690s, Newton wrote,

> [The ancients regarded] a problem as resolved when a geometer had in his own view completed its analysis, and as solved once he had without analysis learnt how to compose it. Whence the solution of problems by the construction of an equation would, to the ancients' mind, seem to be excluded from pure geometry, unless perhaps insofar as an algebraist who is less cognizant of geometry should propose this particular problem: *To denote the root of a proposed equation geometrically*, or insofar as a geometer should gather from the construction of an equation a solution of a kind propoundable and demonstrable without knowledge of the equation.[37]

Newton often insisted on the fact that geometry solves problems in a simpler and more economical way. For instance, in a treatise written in the early 1680s and entitled "Geometria Curvilinea," Newton wrote,

> Men of recent times, eager to add to the discoveries of the ancients, have united specious arithmetic [i.e., algebra] with geometry. Benefiting from that, progress has been broad and far-reaching if your eye is on the profuseness of output but the advance is less of a blessing if you look at the complexity of its conclusions. For these computations, progressing by means of arithmetical operations alone, very often express in an intolerably roundabout way quantities which in geometry are designated by the drawing of a single line.[38]

In the late 1690s, Newton restated these claims when commenting upon Antonio Hugo de Omerique's *Analysis Geometrica* (1698), a work devoted to a laborious geometrical approach to determinate problems (§14.1). Newton became all the more resolute in his classical anti-algebraic approach with the passing of the years. In 1694 he is reported to have stated, "Algebra is the Analysis of the Bunglers in Mathematics."[39]

In sum, according to Newton, Cartesian analysis is often less elegant than geometry; does not reveal an easy way to achieve the synthesis; and leads to the "construction of the equation: a solution excluded from pure geometry to the ancients'

[37] MP, 7, p. 251. "[Veteres] existimantes Problema resolutum esse quando Geometra apud se absolverat Analysin, solutum quando sine Analysi componere didicerat. Unde solutio problematum per constructionem aequationis e Geometria pura, ex veterum sententia, excludenda videtur: nisi forte quatenus Algebraista qui Geometriam minus intelligit proponat hoc ipsum problema, *Radicem propositae aequationis Geometrice designare*; aut quatenus Geometra ex constructione aequationis colligat ejusmodi solutionem quae sine aequationis notitia proponi ac demonstrari potest." MP, 7, p. 250.

[38] MP, 4, p. 421. "Nuperi veterum inventis addere studentes, Arithmeticam speciosam conjunxerunt cum Geometria. Ejus beneficio longe lateque progressum est, si copiam rerum spectes, sed minus commode si perplexitatem conclusionum. Nam haec computa per operationes Arithmeticas solummodo progressa, saepissime per ambages haud ferendas exprimunt quantitates quae in Geometriâ ductu unius lineae designantur." MP, 4, pp. 420, 423.

[39] Hiscock, *David Gregory, Isaac Newton and Their Circle* (1937), p. 42.

mind." He criticized Cartesian synthesis for introducing algebraic considerations into composition.

For analysis, Newton believed, geometry is often preferable to algebra. The equation, if used in the analysis, should be neglected in the synthesis; mechanical curves should be accepted in geometrical constructions; and third-degree equations should be constructed by neusis or by intersection of conics different in detail from those of Descartes.

What were Newton's motivations for criticizing a tool, algebra, that he was able to handle so well? Newton's mathematical classicism resonated both with anti-Cartesianism, developed rather early in his career, and later with an admiration for the ancients, which led him to endorse the myth of a prisca sapientia he saw himself as having rediscovered. In the eyes of his acolytes, Newton was a "great Restorer and Improver."[40] While Newton's endorsement of the *prisca* is well attested in manuscripts dating to the 1690s, what prevails in the 1670s is a marked anti-Cartesian attitude, an attitude that spans from optics to matter theory and mathematics. It is in this anti-Cartesian context that Newton developed a great admiration for the geometrical writings of the ancients, while he bitterly criticized the symbolic mathematics pursued by the moderns.[41] Newton's admiration for the ancients deepened as he began searching for a systematic approach to indeterminate problems (see chapter 5).

[40] John Colson to William Jones (September 20, 1737): "I could rather wish that the Geometers of our Age, who certainly do not want Genius, would employ their Talents in cultivating these parts of Science, upon the same principles, and the same foundation upon which they have been so happily settled by our great Restorer and Improver Sir. I. N." Cambridge University Library, Macclesfield Collection, MS Add. 9597.9.19, ff. 96r–96v.

[41] For further information, see Guicciardini, *Reading the Principia* (1999).

5 Against Descartes on Indeterminate Problems

> Whence it comes that a resolution which proceeds by means of appropriate porisms is more suited to composing demonstrations than is common algebra. Through algebra you easily arrive at equations, but always to pass therefrom to the elegant constructions and demonstrations which usually result by means of the method of porisms is not so easy, nor is one's ingenuity and power of invention so greatly exercised and refined in this analysis.
>
> — Isaac Newton, 1693?

5.1 In Search of Ancient Analysis

This chapter considers Newton's criticisms of the Cartesian approach to indeterminate problems. These are problems that, when reduced to algebraic symbolism, typically lead to a polynomial equation in two unknowns x and y. Their solution is a curve whose points with coordinates (x, y) in a Cartesian coordinate system satisfy the equation. Of course, equations in more than two unknowns can also occur. Recall that the process that leads from the statement of the problem to the equation is the analysis (or resolution) of the problem, and the process that leads from the equation to the construction of the solution curve is the synthesis (or composition) of the problem.

In the 1670s, Newton developed a strong conviction that Descartes' canon for indeterminate problems (§3.2.4) was to be rejected. In his opinion, not only was the synthesis of indeterminate problems proposed in the *Géométrie* unsatisfactory, but Cartesian analysis was also inferior to the method of analysis of the ancient geometers.[1]

When comparing the analysis of the moderns to that of the ancients, Newton often referred to aesthetic values, such as elegance and conciseness, which proved the excellence of classical geometry over modern algebra. Newton further claimed that ancient analysis leads to the synthesis, the construction of the curve, in a more straightforward way.

Epigraph from MP, 7, p. 261. "Unde fit ut Resolutio quae per debita Porismata procedit sit aptior componendis demonstrationibus quam Algebra vulgi. Per Algebram facile pervenitur ad aequationes sed inde saepe ad elegantes illas constructiones ac demonstrationes pergere quae per methodum Porismatum prodire solent, non est adeo facile, sed nec ingenium et inventionis vis in hac Analysi tantopere exercetur & excolitur." MP, 7, p. 260.

[1] The ancient analysis had to be reconstructed from the scanty available sources, most notably Pappus's *Collectio* (§3.1).

It is in this context that the Pappus problem became a priority for Newton. This problem was central to Descartes' *Géométrie*. Descartes had presented his algebraic analysis as superior to the ancients' method, succinctly described in the seventh book of Pappus's *Collectio* (§3.1). In the *Géométrie* he proposed a solution to the Pappus problem of three or four lines as a paradigm of the superiority of his method over that of the ancients (§3.4). His method, he claimed, was preferable to the ancient one. Indeed, according to Descartes, neither Euclid nor Apollonius had been able to thoroughly tackle its generalization to n lines. Newton was of a different opinion. In the late 1670s, commenting upon Descartes' solution of the Pappus problem, he stated with vehemence,

> To be sure, their [the Ancients'] method is more elegant by far than the Cartesian one. For he [Descartes] achieved the result [the solution of the Pappus problem] by an algebraic calculus which, when transposed into words (following the practice of the Ancients in their writings), would prove to be so tedious and entangled as to provoke nausea, nor might it be understood. But they accomplished it by certain simple proportions, judging that nothing written in a different style was worthy to be read, and in consequence they were concealing the analysis by which they found their constructions.[2]

The reader of this book, with the benefit of hindsight and training in twenty-first-century mathematics, might consider this Newtonian statement a misunderstanding of the role and strength of Cartesian algebra. Of course, when algebraic symbols are translated into connected prose, they often lead to a rather opaque mathematical demonstration. It might be said that the introduction of symbolism at the beginning of the seventeenth century was proposed by its defenders as a vehicle for freeing mathematical demonstrations from cumbersome verbal formulations. Further, only algebra could allow generalizations unthinkable in geometry—in the case at hand, a streamlined generalization of the Pappus problem to n lines.

---

[2] This statement occurs at the beginning of the manuscript entitled "Veterum Loca Solida Restituta" (§5.3). MP, 4, p. 277. "Imo vero eorum methodus longe elegantior est Cartesiana. Ille rem peregit per calculum Algebraicum qui in verba (pro more Veterum scriptorum) resolutus adeo prolixus et perplexus evaderet ut nauseam crearet nec posset intelligi. At illi rem peregerunt per simplices quasdam Analogias, nihil judicantes lectu dignum quod aliter scriberetur, & proinde celantes Analysin per quam constructiones invenerunt." MP, 4, p. 276. I have slightly altered Whiteside's translation. Newton was not alone in his battle against the algebraists. Similar statements can be found in the polemical works of Thomas Hobbes. For instance, Hobbes criticized the algebraist Wallis with the following words: "[Y]ou show me how you could demonstrate the ... articles a shorter way. But though there be your symbols, yet no man is obliged to take them for demonstration. And though they be granted to be dumb demonstrations, yet when they are taught to speak as they ought to do, they will be longer demonstrations than these of mine." Hobbes, *The English Works* (1839–45), 7, pp. 281–2. Probably the person who had the greatest influence on Newton in this respect was Barrow (§8.1).

However, one should not underestimate the values that informed Newton's opposition to Cartesian algebra. The invectives against the use of algebraic symbols that express Newton's opposition to the Leibnizian calculus must be viewed as part of a larger project that Newton had in mind: restoring the geometrical analysis of the ancients.[3] This project led Newton to develop important ideas in projective geometry. I believe that Newton was pursuing this program with the conviction that he was both a follower of the ancients and an innovator, that is, he conceived himself to be a creative mathematician who, following the practice of the ancients, was able to contribute new results in that venerated tradition. Reading Newton's defense of geometry as a backward move and identifying algebraization as a progressive element in seventeenth-century mathematics does not capture the values that underlay the confrontation between mathematicians such as Huygens, Barrow, and Newton on the one hand and Descartes, Wallis, and Leibniz on the other.

5.2 Porisms

5.2.1 Reading Pappus's *Collectio*

In the late 1670s, Newton turned to Pappus for instructions: he began reading especially the seventh and eighth books of the *Collectio*.[4] What Newton discovered was a complex text, to which he applied the hermeneutical techniques he had mastered as a biblicist. From this exegetical work he derived a series of suggestions concerning the ancient method of analysis as well as a geometrical solution to the Pappus problem of three or four lines. Historians of Greek mathematics will certainly find Newton's conclusions to be unfamiliar, despite the fact that he was convinced of having penetrated the mysteries behind the ancient method of discovery.

Several aspects of Pappus's *Collectio* must have fascinated Newton. As previously discussed (§3.1), in his seventh book Pappus had introduced the method of analysis and synthesis. Early-modern mathematicians knew many examples of the ancient synthesis epitomized by Euclid's *Elements*: a procedure where one conclusively deduces a consequence (a theorem or a geometrical construction) from given premises. However, from Pappus's text it was difficult to discern what the method

[3] Newton did not note that Descartes had maintained similar ideas on the concealed analysis of the ancients in "Responsio ad Secundas Obiectiones" in *Meditationes de Prima Philosophia* (1641) (AT, 7, pp. 155–6).

[4] MP, 4, pp. 274–55. Whiteside cautiously surmised that Newton might have been influenced by Fermat's "Porismatum Euclidaeorum Renovata Doctrina & Sub Formâ Isagoges Recentioribus Geometris Exhibita," which appeared in *Varia Opera Mathématica* (1679), pp. 116–9. MP, 4, pp. 224, 284, 316, 318, and MP, 7, p. 243. Newton's early acquaintance with Greek analysis might have been derived from reading Frans van Schooten's "Apollonii Pergaei Loca Plana Restituta" in *Exercitationum Mathematicarum Libri Quinque* (1657).

of analysis might have been. After providing a definition of analysis as a path that leads from what one is seeking as if it were already established to something that has already been established by synthesis; and after drawing a distinction between theorematic and problematic analysis, Pappus described a series of works by Apollonius, Euclid, Eratosthenes, and Aristaeus where this method—useful for those "who want to acquire a power in geometry that is capable of solving problems set to them"—was fully explained. Pappus devoted most of Book 7 to a series of lemmas whose purpose was to facilitate the understanding of the works belonging to the "domain of analysis." These included Euclid's *Data* and Apollonius's *Conics*. In Newton's time only the former was accessible in its entirety; of the latter only the first four books were known.[5] The other works referred to by Pappus were not extant: one could try to understand their content through Pappus's brief descriptions and lemmas.

According to the *Collectio*, the highest parts of the method of analysis—understood by Newton and by most of his contemporaries as the "lost method of discovery" of the ancients—were contained in the three lost books of Euclid's *Porisms*, which were described as a "very clever collection for the analysis of more weighty problems."[6] But what is a porism? What are these supposedly heuristic tools which, according to Pappus, "have a delicate and natural aspect, cogent and quite universal, and pleasant for people who know how to see, and how to find?"[7] This is still considered an open question by some historians of Greek mathematics. Robert Simson in the eighteenth century and Michel Chasles in the nineteenth provided detailed reconstructions of Euclid's lost work.[8] From Newton's manuscripts it emerges that he anticipated some of their results. Indeed, Whiteside's edition of *Mathematical Papers* has shown that from the 1670s to the 1690s Newton embarked on a research program aimed at restoring the *Porisms*.

[5] Latin compendia of Books 5–7 were produced by Giovanni Alfonso Borelli and Abraham de l'Echelle in 1661 (Bologna), and by Christian Rau in 1667 (Kiel). Both Borelli's and Rau's works were based on Arabic paraphrases. It was only in 1710 that Halley produced a reliable edition of the *Conics*, 1–7; he based Books 1–4 on a Greek manuscript, and 5–7 on manuscripts in Arabic now held at the Bodleian. Halley's restoration of Book 8 was based on Pappus's lemmas. As for the first four books of the *Conics*, Newton employed Barrow's "edition." In Harrison, *The Library of Isaac Newton* (1978), we find the following works: the first four books of Apollonius's *Conics* in Barrow, *Archimedis Opera* (1675); Halley's edition of Apollonius, *De Sectione Rationis Libri Duo* (1706) and *Apollonii Pergaei Conicorum Libri Octo et Sereni Antissensis De Sectione Cylindri & Coni Libri Duo* (1710).

[6] Pappus, *Book 7 of the Collection* (1986), p. 94.

[7] Ibid.

[8] Simson, *Opera Quaedam Reliqua* (1776). Chasles, *Les Trois Livres de Porismes d'Euclide* (1860). The two enterprises cannot be equated. Simson's work is a genuine attempt to recover Euclid's *Porisms*; Chasles produced a much more theoretical work, which nonetheless is enlightening for historians.

Newton was struck by the idea that porisms were akin to locus problems. As previously mentioned (chapter 3), early-modern mathematicians understood that these problems require the construction of a locus in function of some set of given conditions.[9] Most notably, Newton defined the Pappus problem as a porism.[10]

Newton envisaged a close connection between Euclid's *Data* and *Porisms*.[11] The propositions of the *Data* are concerned with determinate problems and have the form *if A is given, then A' is given.* Newton conceived of porisms as propositions similar, but much more advanced, than the ones occurring in the *Data*, which in some instances state that if a straight line or a curve C is given, then another straight line or curve C' related to the former is also given. Some of Newton's examples of porisms actually have to do with transformations from curve to curve.

Newton also speculated on the possibility that by porisms the ancients might have meant theorems related to the projective properties of conic sections. For him, it was plausible to assume that the ancients were able to discover new theorems and solve new problems on conic sections by identifying properties invariant by central projection. Perspective (and, more generally, projective) transformations are, of course, a tool for associating a given curve to another curve.

Several results presented by Pappus as related to porisms are nowadays best read in terms of cross-ratios of segments in certain configurations of intersecting lines.[12] By using these results, Pappus was able to prove the theorem that still bears his name.[13] Whether the ancients conceived of porisms in the context of some form of prototypal projective geometry is a question that is best addressed in a book devoted to Greek mathematics.[14]

---

[9] See MP, 7, pp. 301, 329. The ancient understanding of the locus problem might have been very different. See Acerbi, "Introduzione" in Euclide, *Tutte le Opere* (2007), pp. 463–82. One of Pappus's definitions of a porism refers to locus problems. It is a definition from which Pappus distanced himself, arguing that it was adopted by moderns on the basis of an accidental trait. This definition nevertheless clearly reveals one property, albeit not an essential one, of porisms: "[A] porism is what is short by a hypothesis of (being) a theorem of a locus. The form of this class of porisms is the loci, and these abound in the *Domain of Analysis.* This kind, separated from the porisms, has been accumulated and handed down because of its being more diffusible than the other forms." See Jones's commentary in Pappus, *Book 7 of the Collection* (1986), pp. 391–2.

[10] MP, 7, p. 399.

[11] "Sunt igitur Euclidis data nihil aliud quam Porismata sed his of simplicitatem inventionis nomen Datorum potius impositum est." MP, 7, p. 262.

[12] Pappus, *Book 7 of the Collection* (1986), pp. 560–2,

[13] Namely, "If A, B, and C are three points on one line, D, E, and F are three points on another line, and AE meets BD at X, AF meets CD at Y, and BF meets CE at Z, then the three points X, Y, and Z are collinear."

[14] See Knorr, *The Ancient Tradition of Geometric Problems* (1993), pp. 108–120 (esp. pp. 116–120). A thorough discussion of porisms is provided in Jones's commentary in Pappus, *Book 7 of the Collection* (1986), pp. 66–70 and 547–72. See also the very interesting Acerbi, "Introduzione" (2007), pp. 733–44.

Newton certainly interpreted porisms in projective terms and thus contributed to a mathematical program thriving in the seventeenth century thanks to the works of Gérard Desargues, Blaise Pascal, and Philippe de La Hire.[15] In his classification of cubic curves Newton relied on perspective transformations in order to go beyond the algebraic boundaries of Cartesian mathematics (see chapter 6). The available evidence indicates that Newton was aware of the fact that the cross-ratio of any four elements of a form is equal to the cross-ratio of the corresponding four elements in any form projectively related to it.[16] Therefore, he studied the *Collectio* and began systematizing Pappus's theorems concerning the invariance of cross-ratios. These researches culminated in the manuscript treatises on the "composition and resolution of the ancient geometers," which Newton wrote in the 1690s (see part V).[17]

In the first pages of Book 7 of the *Collectio*, Pappus presented two porisms (see figures 5.1 and 5.2) that, he claimed, occurred at the beginning of Euclid's lost work.[18] Newton's attention was, of course, caught by these two porisms, which bear some resemblance to his organic method for the construction of conics and higher-order curves. I first consider the so-called hyptios porism proposed by Pappus, which, we are told, summarizes the content of ten propositions in Euclid's first book (§5.2.3). I then turn to the other porism, here referred to as the main porism (§5.2.4). As a preliminary to this discussion of Pappus's porisms it will be useful to recall the definition of cross-ratio.

5.2.2 Cross-ratios

The cross-ratio of four points on a line (in that order) A, B, C, D is defined as

$$(ABCD) = CA/CB : DA/DB, \tag{5.1}$$

where all the segments are thought to be signed (that is, CA is the length of the segment from C to A, etc.). The cross-ratio clearly does not depend on the selected

---

[15] Newton knew La Hire, *Nouvelle Methode en Geometrie pour les Sections des Superficies Coniques et Cylindriques* (1673). Whiteside wrote (MP, 6, p. 271) that the book was bought soon after its publication by the Cambridge University Library; it was reviewed in the *Philosophical Transactions* for March 1676, and was referred to by Hooke in his letter to Newton dated November 24, 1679 (*Correspondence*, 2, p. 298). The extent of Newton's indebtedness to La Hire is, however, unclear.

[16] The invariance of the cross-ratio under a central perspectivity is self-evident and can be extended to a projectivity conceived of as the composition of perspectivities.

[17] See MP, 7, pp. 185–561, and especially "Inventio Porismatum" (pp. 230–47), "Proemium" to "Geometriae Libri Tres" (esp. pp. 260–77), "De Compositione et Resolutione Veterum Geometrarum" (pp. 304–39), where Newton elaborated long lists of porisms taken from Pappus's *Collectio*.

[18] Pappus, *Book 7 of the Collection* (1986), pp. 99–101.

direction of the line $ABCD$ but does depend on the relative position of the points and the order in which they are listed. Although central projection does not preserve distance and ratio between two distances, it does preserve the cross-ratio. That is why this concept is so important in projective geometry.[19]

One can also define the cross-ratio (a, b, c, d) of a pencil of four coplanar straight lines a, b, c, d (the rays) which meet at a point P (called the vertex of the pencil). It is defined as another double ratio, now of sines:

$$(a, b, c, d) = \sin(cPa)/\sin(cPb) : \sin(dPa)/\sin(dPb), \tag{5.2}$$

where angles are also considered signed (in a natural way).

The notation $(P : P_1, P_2, P_3, P_4)$ is also used, where P_i denote four coplanar points, for the cross-ratio of the pencil of four coplanar straight lines PP_1, PP_2, PP_3, PP_4 meeting at P.

It can be proven that if a pencil of four straight lines a, b, c, d meeting at a point P is intersected by any transverse straight line in four points A, B, C, D, then

$$(ABCD) = (a, b, c, d). \tag{5.3}$$

Therefore, the cross-ratio $(ABCD)$ is a constant no matter how the transverse straight line is drawn.

The projective invariance of the cross-ratio of four collinear points leads to consideration of its form when one of the four points lies at infinity. So if $D = \infty$, then

$$(ABC\infty) = CA/CB. \tag{5.4}$$

5.2.3 The Hyptios Porism

The enunciation of this porism (figure 5.1), according to Jones's edition of Pappus, is as follows:

> If the intersections A, B, Γ of three variable straight lines l_1, l_2, l_3 with a straight line l_4 are given, while the intersection of l_2 and l_3 (Δ) lies on a given straight line m_1 and the intersection of l_1 and l_3 (Z) lies on a given straight line m_2, then it is possible to construct a straight line m_3 on which the intersection of l_1 and l_2 (E) lies.[20]

---

[19] Alternative (but equivalent) definitions of cross-ratio can also be found. In fairly old books the term used is anharmonic ratio. Here I follow Casey, *A Treatise on the Analytical Geometry of the Point, Line, Circle, and Conic Sections* (1893), pp. 55, 343. See also Salmon, *A Treatise on Conic Sections* (1896), p. 55. For an accessible presentation, see A. Bogomolny, Cross-Ratio, http://www.cuttheknot.org/pythagoras/Cross-Ratio.shtml, accessed March 17, 2008.

[20] Pappus, *Book 7 of the Collection* (1986), p. 549. The original formulation is in Pappus, *Book 7 of the Collection* (1986), pp. 98–9.

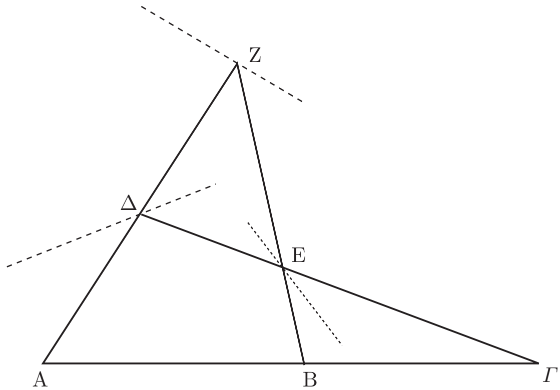


Figure 5.1
Diagram for the hyptios porism, one of the two propositions from *Porisms* reported by Pappus in Book 7 of the *Collectio*. Source: Courtesy of Fabio Acerbi.

So there are four lines that intersect in six points. Three points (A, B, and Γ) are fixed and collinear, two (Δ and Z) are constrained on given straight lines (broken lines in figure 5.1). It is stated that under these conditions a straight line can be constructed such that the other intersection point E will lie on it. Special cases (paryptios cases) arise when two lines are parallel, that is, when one of the intersections is projected to infinity. As Jones observed, this porism is projective in character because it concerns only incidence and collinearity.[21] Jones further made the point that "Euclid did not have the mathematical equipment to treat parallel cases as equivalent to the general case (the "hyptios" porism), and this fact partly explains how there could be several variations (ten propositions) of this porism."[22] Newton, however was quick to embed porisms within the framework of projective geometry. For instance, in the 1690s he studied the hyptios porism in terms of cross-ratios.[23] He further approached it in terms of his kinematic conception of geometrical magnitudes, so that he seemed to suggest that if the points Δ and Z slide on two (broken) straight lines, the corresponding point E describes a (dotted) straight line as well.[24] Projective geometry and

[21] "It is nearly equivalent to the dual of Desargues's theorem, namely that if the corresponding sides of two triangles meet in three collinear points, the lines joining corresponding vertices will intersect in one point." Pappus, *Book 7 of the Collection* (1986), p. 556.
[22] Pappus, *Book 7 of the Collection* (1986), p. 556.
[23] Newton considered the hyptios porism in the "Proemium" to "Geometriae Libri Tres." MP, 7, pp. 268–9.
[24] Newton, however, employed the traditional verbs *tangere* or *contingere*, which do not convey the kinematic meaning of the verb *to trace*, employed in Whiteside's translation. See, e.g., MP, 7, pp. 268, 328.

organic constructions of curves were closely connected in Newton's mathematical practice (§5.4).

5.2.4 The Main Porism

The main porism is as follows (figure 5.2).

> The straight lines ΔE and KZ are given in position. The three points A, B and Z are given too. A ratio a/b is given. The two straight lines $A\Gamma$ and $B\Gamma$ are variable but subject to the following restrictions: (i) each passes through a point (respectively A and B), and (ii) they intersect in a point Γ which lies on ΔE.
>
> One has to construct a straight line $H\Lambda$ and a point H lying on it such that, if Λ is the intersection of this straight line with $B\Gamma$, the segments ΛH and KZ are to each other in the given ratio: $\Lambda H/KZ = a/b$.

The Newtonian manuscript devoted to this porism dates from 1690s.[25] Newton's approach can be summarized as follows. If one considers four different configurations of the variable lines so that they meet in four points Γ_1, Γ_2, Γ_3, Γ_4 on ΔE (and meet the lines in corresponding points K_1, K_2, K_3, K_4 and Λ_1, Λ_2, Λ_3, Λ_4), the cross-ratios of the line pencils with vertices A and B and sides AK_i and $B\Lambda_i$ are equal.

Newton knew that (A: Γ_1, Γ_2, Γ_3, Γ_4) = (A: K_1, K_2, K_3, K_4), since the same pencil cut by two different transversals generates sequences of points with the same cross-ratio. Similarly, (B: Γ_1, Γ_2, Γ_3, Γ_4) = (B: Λ_1, Λ_2, Λ_3, Λ_4). Further, (A: Γ_1, Γ_2, Γ_3, Γ_4)= (B: Γ_1, Γ_2, Γ_3, Γ_4), since the points Γ_i lie on a straight line. Therefore, (A: K_1, K_2, K_3, K_4)= (B: Λ_1, Λ_2, Λ_3, Λ_4).[26]

As Newton observed, the equivalence (A: K_1, K_2, K_3, K_4)= (B: Λ_1, Λ_2, Λ_3, Λ_4) still holds when Γ, instead of lying on a straight line, lies on a conic passing through A and B.[27] Indeed, also for four points Γ_i on such a conic section, the cross-ratios

[25] Entitled "Inventio Porismatum." MP, 7, pp. 242–5.

[26] Given all these equivalences of cross-ratios, in order to solve the main porism Newton chose the straight line $H\Lambda$ so that the intersections K and Λ pass simultaneously to infinity. This condition identifies a direction of $H\Lambda$ and therefore a set of parallel lines. Which line belonging to this set answers the porism? And how to find point H? Consider one of these parallel lines, and mark the points on it with an asterisk. Let the point $H^* = \Lambda_1^*$ correspond to the configuration in which K coincides with $K_1 = Z$; denote this configuration by Γ_1, K_1, Λ_1^*. The equality of cross-ratios just demonstrated can be stated as (K_1, K_2, K_3, ∞)= (Λ_1^*, Λ_2^*, Λ_3^*, ∞), that is, $K_3K_1/K_3K_2 = \Lambda_3^*\Lambda_1^*/\Lambda_3^*\Lambda_2^*$. Therefore a sequence of points Λ_1^*, Λ_2^*, Λ_3^* on any one of the parallel lines just identified, and K_1, K_2, K_3 on the given straight line KZ, is such that the ratios $K_3K_1/\Lambda_3^*\Lambda_1^* = K_3K_2/\Lambda_3^*\Lambda_2^*$ are equal to some constant. In order to solve the porism, this constant must be equal to the given ratio a/b. Of all the parallel lines, the one that satisfies the condition $K_3K_2/\Lambda_3^*\Lambda_2^* = a/b$ must be chosen.

[27] MP, 7, p. 244.

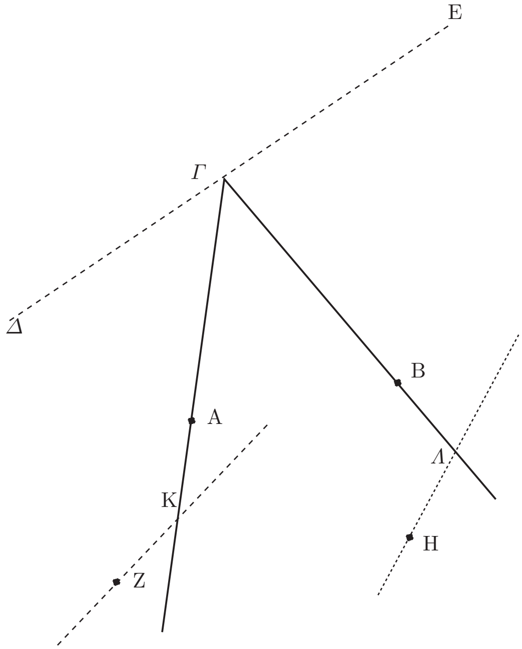


Figure 5.2
Diagram for the main porism reported by Pappus in Book 7 of the *Collectio*. Courtesy of Fabio Acerbi.

(A: Γ_1, Γ_2, Γ_3, Γ_4) and (B: Γ_1, Γ_2, Γ_3, Γ_4) are equal, since the following property is valid.

> **The anharmonic property of conics**: If Γ_1, Γ_2, Γ_3, Γ_4 are four fixed points of a conic, and A is a variable point of the conic, then the cross-ratio (A: Γ_1, Γ_2, Γ_3, Γ_4) is constant.[28]

This property has an important converse:

> **Steiner's theorem** If p and p' describe pencils of lines, with vertices A and B respectively, and if the rays of the two pencils are associated in pairs in such a way that the cross-ratio of any four rays p is equal to the cross-ratio of the corresponding rays p', then the locus of the point of intersection of corresponding rays is a conic through A and B.[29]

[28] Casey, *A Treatise on the Analytical Geometry of the Point, Line, Circle, and Conic Sections* (1893), p. 343; Salmon, *A Treatise on Conic Sections* (1896), p. 252.
[29] Kneebone and Semple, *Algebraic Projective Geometry* (1998), p. 27.

The extension of the main porism to conics, rendered possible by these properties, was probably exploited by Newton in discovering the technique of the organic description of conics (§5.4). The organic generation of conics played an important role in Newton's anti-Cartesian geometrical approach to the Pappus problem of three or four lines.

5.3 Newton's Two-Step Approach to the Pappus Problem

5.3.1 Errors in Descartes' *Géométrie*

A manuscript that according to Whiteside was composed in the late 1670s contains some interesting criticisms aimed against Descartes' pronouncements on the role of the Pappus problem in providing a classification of geometrical curves. The title of this manuscript is "Errores Cartesij Geometriae."[30] After several critiques of Descartes' conceptions of simplicity, Newton considered an idea that is voiced several times in the *Géométrie*, namely, that every geometrical curve is the solution of a Pappus problem of n lines. Descartes hoped that every acceptable (geometric) curve could thus equally be classified according to either the degree of the equation or the number of lines in the pertaining Pappus problem.[31] Indeed, as the number of lines increases, so does the degree of the pertaining equation: for three or four lines the equation is second-degree, for five or six lines the equation is third-degree, and so on.

Newton observed that a sixth-degree curve is defined by

$$\frac{(6+1)(6+2)}{2} - 1 = \frac{6(6+3)}{2} = 27 \tag{5.5}$$

constants. One has simply to consider the number (27) of coefficients occurring in the general polynomial of sixth degree. The Pappus eleven- or twelve–lines locus (a sixth-degree curve) is defined by 25 constants. Indeed, a generalization of equation (3.1) for twelve lines is

$$d_1 d_2 d_3 d_4 d_5 d_6 = k d_7 d_8 d_9 d_{10} d_{11} d_{12}. \tag{5.6}$$

[30] Add. 3961.4, ff. 23r–24r in MP, 4, pp. 336–45. Newton's criticisms relate to the second Latin edition of Descartes' work. A copy signed by Newton and with marginalia in Newton's hand consisting of a few occurrences of catchwords such as error and non Geom was found in 1971 in the Wren Library. MP, 7, p. 194. Some of these marginalia are closely related to "Errores Cartesij Geometriae."

[31] "il n'y a pas une ligne courbe qui tombe sous le calcul & puisse estre receüe en Geometrie, qui n'y soit utile pour quelque nombre de lignes." *Géométrie*, p. 324 [58]. Newton observed: "Erravit praeterea Cartesius in eo quod asseruit omnes curvas quas geometricas vocat utiles esse in Problemate Pappi." MP, 4, p. 340. On Descartes, see Bos, *Redefining Geometrical Exactness* (2001), pp. 281–3, 352–4.

Thus the locus is defined by twelve lines given in position (24 constants) plus a constant k. Newton concluded that there are geometrical curves that are not solutions of a Pappus problem.[32] Therefore, the Pappus problem could not play the role attributed to it by Descartes. It is again the confounding of algebra and geometry that, according to Newton, led Descartes and his followers astray. The Pappus problem was not, in Newton's opinion, subservient to algebraic classification; it was an interesting geometrical problem, and Newton devoted his efforts to its geometrical analysis and synthesis.

5.3.2 Newton's Strategy

Newton's researches on the Pappus problem of three or four lines can be found in two manuscripts dating from the late 1670s and early 1680s: "Veterum Loca Solida Restituta" and "Solutio Problematis Veterum de Loco Solido."[33] The second one is definitely a greater accomplishment, its most important achievements are considered here.

The opening of "Solutio Problematis Veterum de Loco Solido" is particularly interesting because it was incorporated almost verbatim into Section 5, Book 1, of the *Principia*. The solution to the Pappus problem is reached in Corollary 2 to Lemma 19 of the *Principia*, where Newton added a triumphant conclusion:

> And thus there is exhibited in this corollary not a computation but a geometrical synthesis, such as the ancients required, of the classic problem of four lines, which was begun by Euclid and carried on by Apollonius.[34]

Recall that Descartes saw his ability to solve the Pappus problem of three or four lines and extend it to n lines as a clear indication that his own analysis was superior to that of the ancients. Newton aimed to disprove this.

Newton's approach to the Pappus problem of three or four lines in "Solutio Problematis Veterum de Loco Solido" can be divided into two steps:

---

[32] "Latior est itaque natura curvarum hujus ordinis quam quae per Problema Pappi omnimodo designentur." MP, 4, pp. 342–4. A slight error in Newton's reasoning was noted by Whiteside in his commentary, namely, Newton wrote that the twelve lines are determined by 22 constants instead of 24. The best study of "Errores Cartesij Geometriae" and the marginalia pertaining to the *Géométrie* is Galuzzi, "I Marginalia di Newton alla Seconda Edizione Latina della Geometria di Descartes e i Problemi ad Essi Collegati" (1990).

[33] MP, 4, pp. 274–82, 282–321. The critical apparatus provided by Whiteside (see esp. MP, 4, p. 284, note 2) allows the reader to consider the difficult collation necessary to reconstruct these manuscripts. See also "Tractatus de Compositione Locorum Solidorum," which consists of variant drafts to the "Solutio." MP, 4, pp. 322–35.

[34] Newton, *Principles*, p. 485. "Atque ita problematis veterum de quatuor lineis ab Euclide incoepti & ab Apollonio continuati non calculus, sed compositio geometrica, qualem veteres quaerebant, in hoc corollario exhibetur." *Principia*, p. 150.

1. Newton proved that the loci defined by the Pappus problem of three or four lines are conic sections.
2. Newton showed how a conic could be constructed via a mechanism involving rotating rulers (see §5.4). In this step one actually identifies a conic that is the sought solution to the Pappus problem for a particular configuration of three or four lines and a constant k (see equation (3.1)).

5.3.3 First Step

I look briefly here at the first propositions of "Solutio Problematis Veterum de Loco Solido," which allowed Newton to achieve the first step of his solution of the Pappus problem.

Newton made use of a property proved in Propositions 17, 19, 21, and 23 of the third Book of Apollonius's *Conics*, which he explicitly cited.[35] The property states that "if two chords AB and DE of a conic intersect in C, the rectangles contained by their segments are proportional to the squares on the parallel diameters," that is, the ratio $(AC \times CB) : (DC \times CE)$ does not change when the chords AB and DE are moved parallel to themselves (figure 5.3).[36]

In Proposition 1, Newton proved on the basis of this property that if $ABDC$ is a quadrilateral inscribed in a conic (figure 5.4) and from any point P on the conic straight lines PQ, PR, PS, PT are drawn making given angles with the sides AB, CD, AC, BD, then $PQ \times PR$ will be to $PS \times PT$ in a given ratio. [37] This proves that for any point P belonging to a conic Pappus's condition (equation (3.1)) holds.

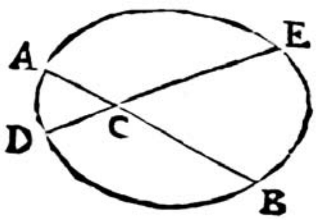


Figure 5.3
Diagram for Proposition 11 in Stirling, *Lineae Tertii Ordinis Neutonianae* (1717), p. 77. Courtesy of the Biblioteca Angelo Mai (Bergamo).

---

[35] *Principia*, p. 144. It is well known that the proof that conics are solutions of the three lines locus problem can be deduced from Propositions, 54, 55, and 56, Book 3, of the *Conics*.
[36] For a discussion of Newton's use of this property in the demonstration of the initial Lemmas 17 and 18, Section 5, Book 1, of the *Principia*, see Milne, "Newton's Contribution to the Geometry of Conics" (1927), p. 102.
[37] This proposition appears in the *Principia* as Lemma 17, Book 1.

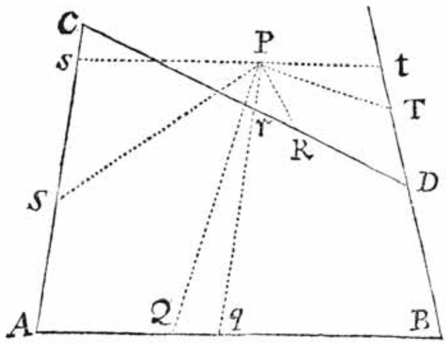


Figure 5.4
Diagram for Lemma 17, Section 5, Book 1, of the *Principia*. Source: Newton, *Philosophiae Naturalis Principia Mathematica* (1726), p. 74. Courtesy of the Biblioteca Angelo Mai (Bergamo).

Proposition 2 proves the converse: if point P moves in such a manner that $PQ \times PR$ is to $PS \times PT$ in a given ratio, then the locus of P is a conic passing through A, B, C, and D.[38]

5.3.4 Second Step

The first step is achieved via rather elementary geometrical methods. Notwithstanding its classical façade, Newton's approach to the second step was guided by an understanding of the fact that conic sections can be defined as those curves that satisfy the "anharmonic property."[39]

From the condition of the Pappus problem of three or four lines (equation (3.1)) it is clear that the required conic will pass through four intersections of the four given lines. It therefore belongs to the sheaf of conics that pass through the vertices of a quadrilateral. Further, a fifth point can be constructed (as Newton showed in Proposition 3). Hence, the problem is reduced to the construction of a conic that will pass through five given points.[40]

In the eighth book of the *Collectio*, Pappus had provided a method for constructing an ellipse that passes through five given points. Newton saw this as positive

[38] This is Lemma 18, Book 1, in the *Principia*.
[39] For a discussion of Newton's solution, see Whiteside's commentary in MP, 4, pp. 275–6, and Di Sieno and Galuzzi, "La Quinta Sezione del Primo Libro dei *Principia*" (1989).
[40] MP, 4, pp. 294ff. Cases in which three points are collinear degenerate into two straight lines.

proof that the ancients could tackle the second step required for a solution of the Pappus problem. In "Veterum Loca Solida Restituta" he wrote,

> Descartes in regard to his accomplishment of this problem makes a great show as if he had achieved something so earnestly sought after by the Ancients and for whose sake he considers that Apollonius wrote his books on conics. With all respect to so great a man I should have believed that this topic remained not at all a mystery to the Ancients. For Pappus informs us of a method for drawing an ellipse through five given points and the reasoning is the same in the case the other conics. And if the Ancients knew how to draw a conic through five given points, does any one not see that they found out the composition of the solid locus? ... To reveal that this topic was no mystery to them, I shall attempt to restore their discovery by following in the steps of Pappus' problem.[41]

As a matter of fact, Descartes never claimed that the ancients could not solve the three or four lines locus. From what he wrote in the *Géométrie* and from the passages of the *Collectio* he discussed, it is clear that Descartes credited Apollonius with a solution to the problem of three or four lines.[42] What Descartes claimed was that his algebraic solution was more satisfactory and that it could pave the way to a systematic generalization of the problem of n lines. Newton seems to have misinterpreted Descartes' statements.

To summarize: Once it is known (first step) that the locus defined by the Pappus problem of three or four lines is a conic, the next step is to construct the conic. The conic is known to pass through five given points. The question then is, How can a conic passing through five given points be constructed? It is by thinking about this problem that Newton came up with several interesting ideas concerning the relations between algebra and geometry, and between the ancient and modern methods of analysis and synthesis.

5.4 Organic Description of Conics

5.4.1 Proposition 7

In Proposition 7 of "Solutio Problematis Veterum de Loco Solido," Newton constructed a conic that passes through five given points, thanks to an organic description that he had obtained in the late 1660s, probably inspired by de Witt and van

[41] MP, 4, pp. 275, 277. "Cartesius de hujus Problematis confectione se jactitat quasi aliquid praestitisset a Veteribus tantopere quaesitum, cujus gratia putat Apollonium libros suos de Conicis sectionibus scripsisse. Sed cum tanti viri pace rem Veteres neutiquam latuisse crediderim. Docet enim Pappus modum ducendi Ellipsin per quinque data puncta et eadem est ratio in caeteris Con. sect. Et si Veteres norint ducere Conicam sectionem per quinque data puncta, quis non videt eos cognovisse compositionem loci solidi. ... Ut pateat hanc rem eos non latuisse, conabor inventum restituere insistendo vestigijs Problematis Pappiani." MP, 4, pp. 274, 276.
[42] Descartes, *Géométrie*, pp. 304–8.

Schooten.[43] As mentioned, this construction is very similar to the main porism, which (as Newton was well aware) can be extended to conic sections (§5.2.4). The organic description was incorporated in the *Principia* as Lemma 21 and Proposition 22, Book 1.

Briefly stated, Newton's organic description can be presented by noting that if two angles of given magnitudes turn about their respective vertices (the poles) in such a way that the point of intersection of one pair of arms always lies on a straight line (the describing line), the point of intersection of the other pair of arms will describe a conic (the describend curve). The proof is discussed in §5.4.4. First, I turn to Newton's early presentation of this technique to John Collins. Newton showed Collins how a conic passing through five given points can be constructed by appropriately choosing the poles, the two angles, and the describing line.

5.4.2 Newton to Collins, August 20, 1672

In a letter dated August 20, 1672, addressed to John Collins, Newton explained his construction (see figure 5.5):

> Let the five points be *A, B, C, D, & E* any three of wch as *A, B & C* join to make a rectilinear triangle *ABC*, to any two angles of wch as *A & B* apply two sectors, their poles as ye angular points, & their leggs to the sides of ye triangle. And so dispose them that they may turne freely about their poles *A & B* without varying the angles they are thus set at. Which done, apply to ye other two points *D & E* successively the two leggs *PQ & RS* wch were before applyed to *C* (wch leggs for distinction sake may be called their describing leggs & the other to *MN & TV* wch were applyed to *AB*, their directing leggs,) & marke the intersections of their directing leggs, wch intersection suppose to be *F* when ye application was made to *D & G* when made to *E*. Draw the right line *FG* & produce it infinitely both ways. And then if you move the rulers in such manner that their directing leggs doe continually intersect one another at the line *GF*, the intersection of their other leggs shall describe the conic section wch will pass through all the said five given points. If three of the given points lye in the same straight line tis impossible far any conick section to pass through them all, And in that case you shall have instead thereof two streight lines.[44]

Most probably Newton conceived of this organic technique as being consonant with several prescriptions about porisms given by Pappus. Indeed, Newton deployed his

[43] See MP, 2, pp. 106–59, for Newton's early researches on organic descriptions, esp. "De Modo Describendi Conicas Sectiones et Curvas Trium Dimensionum Quando Sint Primi Gradus," (pp. 134–51) and MP, 4, pp. 298–303. As Whiteside noted, similar constructions were published by Torricellii and Fermat. MP, 4, pp. 292–3n. Newton, however, according to Whiteside, derived inspiration from Jan de Witt, "Elementa Curvarum Linearum" in appendix to the second Latin edition of Descartes' *Geometria*, pp. 229–38, and van Schooten, *De Organica Conicarum Sectionum in Plano Descriptione Tractatus* (1646), which Newton read in the *Exercitationum Mathematicarum* (1657), pp. 293–368. MP, 1, pp. 34, 40.

[44] *Correspondence*, 2, pp. 230–1.

organic description of conics in the manuscripts concerned with the restoration of ancient Greek porismatic analysis.

The projective character of Newton's organic description is spelled out in Propositions 5 and 6 of "Solutio Problematis Veterum de Loco Solido."[45] These propositions form the theoretical basis of Newton's organic description, that is, they prove, by invoking the anharmonic property, that the locus traced by the intersection of the describing legs is a conic.

5.4.3 From Paper-Tools to Mechanical Tools

The drawing accompanying the letter to Collins (see figure 5.5) is quite realistic and suggests that Newton actually made use of real instruments to trace curves.[46]

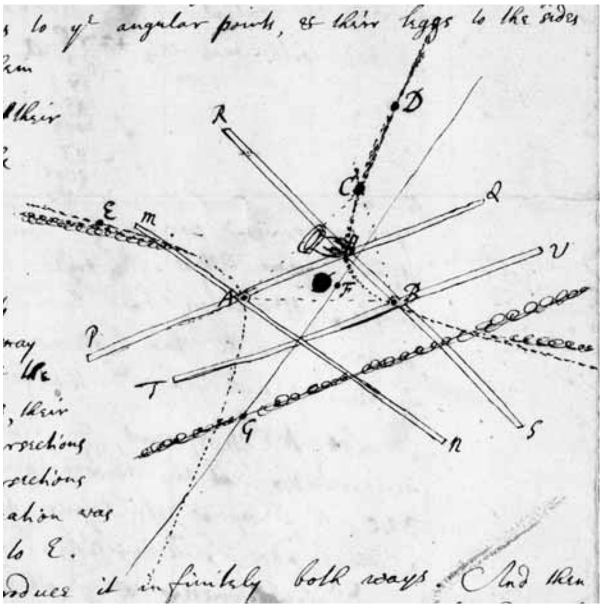


Figure 5.5
Organic construction of a conic through five given points. Note that the conic, in this case a hyperbola, passes through the five given points *A, B, C, D, E*. From a letter of Newton to John Collins (August 20, 1672). Source: Add. 3977.10, f.1v. Reproduced by kind permission of the Syndics of Cambridge University Library.

---

[45] MP, 4, pp. 294–305. These propositions correspond to Lemma 20 and Proposition 22, Book 1, of the *Principia*.

[46] On the use of mechanical instruments as cognitive devices in the seventeenth century, see Bertoloni Meli, *Thinking with Objects* (2006).

This rather irreverent hypothesis is supported by evidence taken from the manuscripts on organics that Newton probably wrote in the late 1660s:

> Two rules ... are to be manufactured so that their legs ... can be inclined to each other, at will, in any given angle ... . And at the junctures there should be a steel pin-point around which the rules may be rotated while the pin is fixed on some given point as its centre. To be sure, the steel nail by which the legs of the sector are joined might be finely sharpened at one end, and on he other threaded to take a nut more or less tightly (as the need arises) which will clamp the legs of the sector in the given angle.[47]

The manuscript continues with a long list of unproven results in which Newton considered the transformation of several describing curves into corresponding describend curves. All these results were merely stated, not demonstrated. Newton's theorems on organic transformations would nowadays be expressed in terms of the theory of birational correspondences of second degree, deploying algebraic tools that Newton did not have at his disposal.

In the *Enumeratio*, Newton summarized some of his results.[48] He stated that if the intersection of the directing legs moves along a conic through A (the describing curve), the intersection of the other legs will trace a cubic through B with a node at A (the describend curve). If the intersection of the directing legs moves along a conic in general, then the intersection of the other legs will trace a cubic or a quartic, in the latter case with nodes at B and A. In the *Enumeratio*, Newton showed how his organic generation can be used to construct a nodal cubic that passes through seven given points.[49]

Commentators, like Whiteside and Shkolenok have asked how Newton could have achieved such extraordinary results on curve transformations via the organic correspondence between the describing and describend curves. Both scholars conclude that Newton's technique must have been graphical.[50] By tracing the describend curve for a given describing curve, Newton could identify the singularities and estimate the curve degree by the maximum number of its meets with a straight line.

---

[47] MP, 2, p. 135.

[48] Newton's work on the classification of cubic curves is discussed in chapter 6.

[49] The organic description of conics was published by Newton in *Arithmetica Universalis*, in the *Enumeratio*, and in the *Principia*; he was clearly quite pleased with it. See Newton, *Mathematical Works* (1967), 2, pp. 95–6; *Principles*, pp. 486–90; MP, 5, pp. 304–5; Turnbull, *The Mathematical Discoveries of Newton* (1945), p. 54.

[50] "A natural question arising after close investigation of Newton's manuscripts on the organic description is, how did he obtain such a great number of correct results at a time when algebraic geometry was practically non-existent?" Shkolenok, "Geometrical Constructions Equivalent to Non-Linear Algebraic Transformations of the Plane in Newton's Early Papers" (1972), p. 36, and Whiteside's note 8 in MP, 2, pp. 107–8.

I would like to surmise that Newton actually made use of a real instrument, as several passages in his manuscripts suggest.[51]

The Pappian tradition in the geometrical theory of organic descriptions does not rule out the possibility that its proponents might have been in contact with instrument makers who actually built real curve-tracing devices. Frans van Schooten, from whose *De Organica Conicarum Sectionum in Plano Descriptione Tractatus* (1646) Newton drew inspiration, explicitly referred, in the subtitle of his work, to the usefulness of this theory to opticians, mechanics, and land surveyors.[52] Even Descartes did not shrink from providing a detailed description of curve-tracing devices and lens-grinding machinery based on his theory of ovals, topics that he extensively discussed with high-ranking men of letters like Constantijn Huygens and humble artisans like Jean Ferrier.

5.4.4 Newton's Proof of the Organic Description of Conics

As stated, Newton did not provide proof for most of his statements on the organic description of curves. However, he did have a proof of the fact that if the describing curve is a straight line, then the describend curve is a conic section, that is, he could prove the organic description that in 1672 he communicated to Collins.

Newton provided such a demonstration in Proposition 7 of "Solutio Problematis Veterum de Loco Solido" (= Lemma 21, Book 1, of the *Principia*) (figure 5.6). The similarities with the techniques of Pappus's main porism are evident. Newton actually conceived of his organic generation of conics as a technique belonging to the ancient porismatic analysis, an interpretation that might strike historians of Greek mathematics as rather strange.

Newton wrote,

> If two movable and infinite straight lines BM and CM, drawn through given points B and C as poles, describe by their meeting-point M a third straight line MN given in position, and if two other infinite straight lines BD and CD are drawn making given angles MBD and MCD with the first two lines at those given points B and C; then I say that the point D, where these two lines BD and CD meet, will describe a conic passing through points B and C. And conversely, if the point D, where the straight lines BD and CD meet, describes a conic passing through the given points B, C and A, and the angle DBM is always equal to the given angle

[51] For instance, Newton provided very concrete directions such as, "After you have described the curve one way, you may test in others whether the description is accurate: precisely by fixing the rule in other angles or by taking other points for the poles of the rulers." MP, 2, p. 121.

[52] The full title is *De Organica Conicarum Sectionum in Plano Descriptione Tractatus: Geometris, Opticis, Praesertim Vero Gnomonicis & Mechanicis Utilis* (1646).

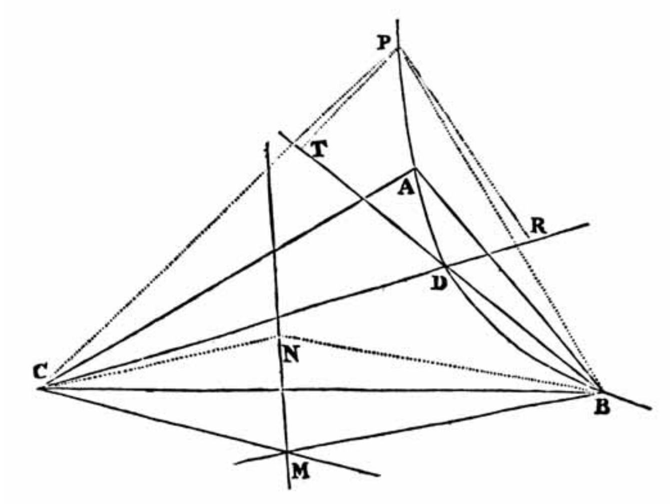


Figure 5.6

Diagram for Lemma 21, Book 1, of the *Principia*. Source: Newton, *Philosophiae Naturalis Principia Mathematica* (1726), p. 80. Courtesy of the Biblioteca Angelo Mai (Bergamo).

> ABC, and the angle DCM is always equal to the given angle ACB; then the point M will lie in a straight line given in position.[53]

Note that Proposition 7 (= Lemma 21) affirms not only that this mechanism generates a conic but that all conics can be generated by such a mechanism.

First, Newton noted that when the directing legs intersect in M, the describing legs intersect in D. Second, when the intersection moves along the straight line from M to N, the other point of intersection will move from D to P. Therefore $\widehat{MBD} = \widehat{NBP}$ and $\widehat{MCD} = \widehat{NCP}$. Let the points R on the straight line CD and T on the straight line BD be such that $\widehat{BPT} = \widehat{BNM}$ and $\widehat{CPR} = \widehat{CNM}$. The following equation immediately follows:

$$\widehat{MBD} - \widehat{NBD} = \widehat{NBP} - \widehat{NBD}; \tag{5.7}$$

therefore

$$\widehat{MBN} = \widehat{PBD} = \widehat{PBT}. \tag{5.8}$$

Also,

$$\widehat{MCD} - \widehat{NCD} = \widehat{NCP} - \widehat{NCD}; \tag{5.9}$$

[53] *Principles*, pp. 486–7. For the reader's convenience I cite Cohen and Whitman's English translation of the *Principia*. This is taken almost verbatim from Proposition 7 of "Solutio Problematis Veterum de Loco Solido." MP, 4, p. 298.

therefore

$$\widehat{NCM} = \widehat{PCR}. \tag{5.10}$$

Since $\widehat{BTP} = \widehat{BNM}$ and $\widehat{MBN} = \widehat{PBT}$, the triangles MBN and PBT are similar. Also, the triangles NCM and PCR are similar. These similarities allow us to state that

$$\frac{PT}{NM} = \frac{PB}{NB} \tag{5.11}$$

and

$$\frac{PR}{NM} = \frac{PC}{NC}. \tag{5.12}$$

But points B, C, N, and P are "stationary"; therefore "PT and PR have a given ratio to each other," or using somewhat more modern algebraic notation,

$$\frac{PT}{PR} = \frac{PB \times NC}{NB \times PC} = k, \tag{5.13}$$

k constant. The constancy of the ratio PT/PR is the "symptom" that allowed Newton to state that the locus traced by the describing legs is a conic. Indeed, in Proposition 5 (= Lemma 20) Newton proved that equation (5.13) holds for all and only for conic sections. He showed that Pappus's condition (equation (3.1)) is equivalent to equation (5.13).

A nineteenth-century proof of Newton's organic description of conics is provided by Salmon (figure 5.7).

5.4.5 Organic Descriptions and Projective Geometry

Several commentators have interpreted the result achieved in Propositions 5 and 7 as a proof of Steiner's theorem. Recall that the anharmonic property states that the pencils from B and C to four points on a conic have the same cross-ratio. In order to examine this I now turn to Proposition 6.[54]

In this proposition Newton showed that if five points are considered on a given conic A, B, C, P, D, a sixth point d can be constructed (figure 5.8).

After tracing AB and AC a parallelogram $ASPQ$ is drawn by tracing PQ parallel to AC and PS parallel to AB. Projecting D from B on PS one obtains point T. Projecting D from C on PQ one obtains point R. If tr is traced parallel to TR one obtains the two segments Pt and Pr, so that

$$\frac{PR}{PT} = \frac{Pr}{Pt}. \tag{5.14}$$

[54] This corresponds to Proposition 22, Book 1, of the *Principia*.

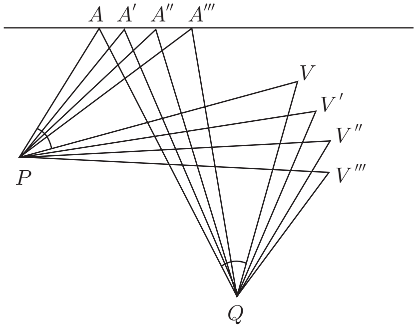


Figure 5.7

A nineteenth-century proof by Salmon of Newton's organic description of conics. Two angles of fixed magnitude rotate about the fixed points P and Q. The intersection of two of their sides traverses the straight line AA'. The construction implies that the intersection of the other two sides traces a curve passing through P and Q. Salmon considered four positions of the angles and four corresponding points, V', V'', V''', V'''', on the curve. He proved that the pencils from P and Q to the four points V', V'', V''', V'''' have the same cross-ratio. Taking four positions of the legs, (i) $(P : A', A'', A''', A'''') = (Q : A', A'', A''', A'''')$, where, as usual, the notation $(P : A', A'', A''', A'''')$ means the cross-ratio of the line pencil with vertex P and sides PA', PA'', PA''', PA''''. Equation (i) holds because the two pencils intersect in four collinear points. Further, (ii) $(P : A', A'', A''', A'''') = (P : V', V'', V''', V'''')$, and (iii) $(Q : A', A'', A''', A'''') = (Q : V', V'', V''', V'''')$ because the angles of the pencils are the same. Therefore, (iv) $(P : V', V'', V''', V'''') = (Q : V', V'', V''', V'''')$. Salmon, *A Treatise on Conic Sections* (1896), p. 300. Source: By Compomat, s.r.l. ©Niccolò Guicciardini.

This implies that

$$\frac{PR}{Pr} = \frac{PT}{Pt} \tag{5.15}$$

and

$$\frac{PR}{rR} = \frac{PT}{tT}. \tag{5.16}$$

The intersection of Cr and Bt determines a new point d which, as Newton stated in Proposition 6, will also be on the conic because of the "symptom" (equation 5.13). Di Sieno and Galuzzi observed that the pencils from the vertices B and C to the four points P, d, D, A have equal cross-ratios,[55] since

$$(B : P, d, D, A) = (P, t, T, \infty) = \frac{PT}{tT} = \frac{PR}{rR} = (P, r, R, \infty) = (C : P, d, D, A). \tag{5.17}$$

---

[55] Indeed, according to equation (5.4), $(P, t, T, \infty) = TP/Tt = PT/tT$ and $(P, r, R, \infty) = RP/Rr = PR/rR$.

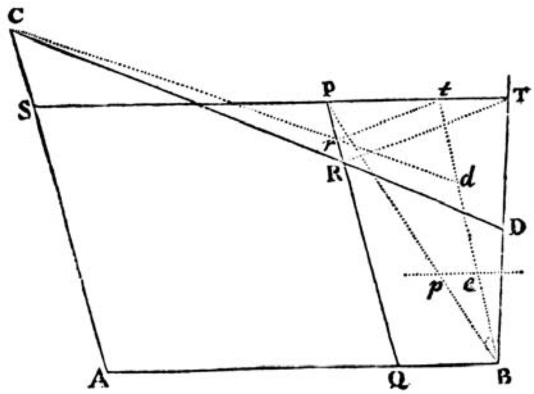


Figure 5.8
Diagram for Proposition 22, Book 1, of the *Principia*. Source: Newton, *Philosophiae Naturalis Principia Mathematica* (1726), p. 82. Courtesy of the Biblioteca Angelo Mai (Bergamo).

This interpretation, which is shared by Whiteside, attributes to Newton an understanding of Steiner's theorem as the foundation of the propositions and organic descriptions presented in "Solutio Problematis Veterum de Loco Solido" and in Section 5, Book 1, of the *Principia*.[56]

5.4.6 Newton's Evaluation of Organic Descriptions

It is interesting to note why the organic description of conics was understood by Newton as part of an analytical method alternative to the Cartesian one.

It is a method alternative to Descartes' because it does not involve any calculation. As Newton wrote,

> These and the following descriptions are of greatest use in determining solid loci [conics] and so on. Precisely, given merely five points [a]and that the curve be a conic[a] *without any preparatory calculation* or knowing the vertex, axis, diameters, center and species of the curve, provided only that its kind is given (that it is a conic) you should even so be able to describe it.[57]

[56] See Whiteside's commentary in MP, 4, pp. 275–6, and Di Sieno and Galuzzi, "La Quinta Sezione del Primo Libro dei *Principia*" (1989). See also Shkolenok, "Geometrical Constructions Equivalent to Non-Linear Algebraic Transformations of the Plane in Newton's Early Papers" (1972).

[57] MP, 2, p. 121. *aa* canceled. Italics supplied. "Haec et sequentes descriptiones plurimum habent usum in locis solidis determinandis, & c. Nemque ex datis tantum 5 punctis [a]& quod curva sit Con sect[a] sine aliquo calculo praemisso, licet ignorantur vertex, axis, Diamteri, centrum et species curvae modo datur genus quod sit conica sectio possis tamen curvam describere." MP, 2, p. 120.

This quotation is taken from a manuscript written, according to Whiteside, in 1667 or 1668. From the very beginning of his research Newton was greatly interested in the possibility of tackling problems "without calculation." Work on curves conceived of as generated by motion was an important aspect of Newton's effort to study curves without the use of symbols.

Newton's method is analytical because it resolves the locus problem by beginning with the assumption that the sought conic passing through the points A, B, C, D, and E has been constructed (see figure 5.5).[58] The elements assumed as given in the problem, namely, the five given points, determine the poles A and B, the angles CBA and CAB, and the straight line GF. This deduction thus provides all the elements that are necessary for the synthesis or composition, namely, the construction of the curve.

The synthesis, the composition of the locus, now simply consists in turning the rulers so that the intersection of the directing legs slides along GF. As Newton wrote in the 1690s,

> Whence it comes that a resolution which proceeds by means of appropriate porisms is more suited to composing demonstrations than is common algebra. Through algebra you easily arrive at equations, but always to pass therefrom to the elegant constructions and demonstrations which usually result by means of the method of porisms is not so easy, nor is one's ingenuity and power of invention so greatly exercised and refined in this analysis.[59]

In this passage, Newton expressed one of his favorite ideas: the notion that it is geometrical analysis that reveals the ingenuity and power of invention of a mathematician, whereas algebra is within the reach of anyone (even a bungler, as Newton is claimed to have said) who is able to manipulate symbols mechanically. Further, while Cartesian analysis leads to an equation in two unknowns from which it is difficult to geometrically construct the curve, porismatic analysis opens the way for an elegant organic description of the solution curve.

Newton highly valued the fact that geometrical analysis can lead to synthesis in a very "easy" and "elegant" way, by simple inversion of the "sequence of argument."[60] But how should curves be constructed? What were Newton's ideas concerning the

[58] Newton's understanding of the analysis of locus problems might greatly have differed from the one adopted in antiquity. See Acerbi, "Introduzione," pp. 463–82.

[59] MP, 7, p. 261. "Unde fit ut Resolutio quae per debita Porismata procedit sit aptior componendis demonstrationibus quam Algebra vulgi. Per Algebram facile pervenitur ad aequationes sed inde saepe ad elegantes illas constructiones ac demonstrationes pergere quae per methodum Porismatum prodire solent, non est adeo facile, sed nec ingenium et inventionis vis in hac Analysi tantopere exercetur & excolitur." MP, 7, p. 260.

[60] "Where, however, the things sought would not easily ensue from the givens, they [the Ancients] either looked for lemmas or porisms through which some new given might be gatherable, or assumed unknowns as givens so that thereby they might gather some given as though it were unknown, and at length by inverting the sequence of argument deduce the things sought from

legitimate techniques for the construction of curves? Newton rejected not only the Cartesian analysis of locus problems but also the Cartesian synthesis. Newton found all means for curve construction proposed in the *Géométrie* to be unacceptable.

5.4.7 Postulates on the Construction of Curves

Newton wrote at length on the construction of curves. What he has to say about this topic is invariably anti-Cartesian. When in the 1690s he reconsidered and systematized his geometrical researches into a multipartite treatise, "Geometriae Libri Duo," he attempted (in a draft of the second book) to define postulates for the organic description of curves. His aim was to distinguish between admissible and "spurious" postulates: "The ancients received into geometry those lines alone which are describable by means of geometric postulates."[61]

The postulates of Euclid's *Elements* allow constructions by straight-edge and compass. Next, Newton notes that more recent authors had added another postulate, which allows the section of a cone by a plane and therefore the generation of conic sections. This postulate does not win the favor of Newton, who believed it had been introduced by later authors who corrupted the style of the ancients:

> But those who afterwards added the postulate on the generating of a line by the section of a cone were not so mindful of human utility; for, because of its difficult mechanical accomplishment, that section is a barren speculation having nothing in joint with the uses of men.[62]

It would be desirable, he thought, to have another postulate "easier in its mechanical execution" which could allow the generation of conic sections and "further curves." But he warned, "Lest we postulates anything which is not legitimate, we need first to exclude spurious postulates."[63]

Which are the "spurious" postulates? Newton, always the arch anti-Cartesian, enumerated and rejected all the curve-tracing methods contemplated in the *Géométrie*. Cone sections had been already excluded. Point-wise constructions were to be rejected (because one has to complete the curve by "a chance of the

---

whatever relationship between the given and the sought." MP, 8, p. 445. "Ubi vero Quaesitum ex datis non facile consequeretur, vel quaerebant Lemmata aut Porismata per quae Datum aliquod novum colligere possent vel assumebant ignota tanquam data ut inde datum aliquod tanquam ignotum colligerent, ac tandem ex relatione quacumque inter data et quaesitum invertendo ordinem argumentationis quaesitum deducerent." MP, 8, p. 444.

[61] MP, 7, p. 383. "Veteres in Geometriam lineas solas receperunt quae per Postulata Geometrica descibi possunt." MP, 7, p. 382. "Geometriae Libri Duo" is discussed in §14.1.

[62] MP, 7, p. 383. "Attamen qui postea postulatum addiderunt de generatione lineae per sectionem Coni haud usibus humanis consulerunt. Nam sectio illa ob difficilem praxin mechanicam nuda est speculatio, nihil habens cum usibus humanis conjunctum." MP, 7, p. 382.

[63] MP, 7, p. 385. "Se ne quid non legitimum postulemus, prius excludenda sunt postulata spuria." MP, 7, p. 384.

hand"), as well as those that "allude to stretched threads" (Newton took the example of the Cartesian oval (figure 3.4).) Finally, he did not accept constructions based on the postulate according to which "given curves" may move in any assigned manner so that "by their intersection fresh curves are described," another Cartesian technique yet one that looks very similar to the organic generation of curves promoted by Newton. There is often something illogical in Newton's anti-Cartesian invectives.

Newton continued the introduction of the second book of his treatise on geometry by proposing those mechanical constructions that employ moving rulers as legitimate. He characterized his organic description of conics by intersection of rotating legs (as well as the organic description of the cissoid; see figure 4.5) and surmised that this is the only legitimate construction of curves. Other organic generations involving rulers, like the Cartesian mesolabum (see figure 3.3), lack generality because not all algebraic curves can be generated by them.[64] Newton surmised that his organic descriptions, by contrast, allow the generation not only of conic sections but also of all curves of higher degree. He observed that if the directing legs move along a conic, the generating legs will describe a higher-order curve (a cubic or a quartic). He stated that by moving the intersection of the directing legs along curves of higher degree the describing legs could generate curves of still higher degree.[65] Recall that the ancient method of porisms was seen by Newton as a technique allowing transformation of one curve into another, so that if one curve is given, the other is given as well. He added the following observation:

> But these descriptions, insofar as they are achieved by manufactured instruments, are mechanical; insofar, however, as they are understood to be accomplished by the geometrical lines which the rulers in the instruments represent, they are exactly those which we embrace ... as geometrical.[66]

Part V goes into further detail on Newton's view of the relation between geometry and mechanics.

5.4.8 Comparing Methods

An interesting comparison between the algebraic and the geometrical approaches to locus problems can be found in *Lucasian Lectures on Algebra*, at the end of the long section entitled "How Geometrical Questions are to be Reduced to an Equation." Clearly, Newton gave great importance to this section, as is shown by its length

---

[64] See figure 3.3 caption.

[65] For a proof, see Miller, "Newton, Aufzählung der Linien dritter Ordnung" (1953), pp. 30–1.

[66] MP, 7, p. 393. "Hae autem descriptiones quatenus per Organa manufacta perficiuntur mechanicae sunt: quatenus vero per lineas geometricas quas organorum regulae representant subintelliguntur fieri, eae ipsae sunt quas ... ut geometricas amplectimur." MP, 7, p. 392.

and originality. What is interesting here is that at the end Newton advances some critical remarks on the use of algebra in the analysis of indeterminate problems. Such statements have been somewhat of a puzzle until recently, when thanks to the publication of Newton's *Mathematical Papers*, the extent of Newton's involvement in a program alternative to the Cartesian one has emerged.

In Problem 53 of *Lucasian Lectures on Algebra* (= Problem 57 of *Arithmetica Universalis*) Newton provided a proof, framed in Cartesian algebraic terms, of his organic description of conics. He proved that the locus traced by the describing legs is a conic section, since it is defined by a second-degree equation. The calculation is heavy compared with the proof in terms of projective geometry.[67]

In the final problems considered by Newton in his *Lectures*, namely, the problems numbered 59, 60, and 61 in the printed *Arithmetica*, we find a comparison between algebraic and geometrical analyses of indeterminate problems. Newton provided two resolutions of these problems: one geometrical, "by means of certain theorems by Apollonius," the other "by algebra alone," according to "Descartes' method."[68] He wrote,

> [C]ertain things which came to me as I wrote I have also intermixed without algebraic solution in order to convey the point that in problems which at first glance seem difficult there is no need always to have recourse to algebra.[69]

Problem 59 in *Arithmetica Universalis* is the one just considered. It requires to find a conic passing through five given points.[70] The geometrical solution of this problem, achieved in Proposition 7 of "Solutio Problematis Veterum de Loco Solido" in terms of the organic description of conics, was highly significant from Newton's point of view; it was the culmination of research he had pursued in order to restore the geometrical analysis of the ancients. As noted, the solution of Problem 59 was a component in the approach to the Pappus problem that Newton had devised in order to demonstrate the superiority of ancient analysis over that of Descartes.

Newton's Cartesian solution of Problem 59 is as follows. He chose the coordinate axes so that the general equation of the conic has the form $a + bx + dy + cx^2 + exy +$

---

[67] See MP, 5, p. 305. Newton attempted to prove this result algebraically in the late 1660s but with no success. See MP, 2, pp. 152–5. An elegant proof can be found in Miller, "Newton, Aufzählung der Linien dritter Ordnung" (1953), pp. 30–1.

[68] MP, 5, pp. 315, 317.

[69] MP, 5, p. 337. "aliqua quae inter scribendum occurebant immiscui sine Algebra soluta, ut insinuarem in problematis quae prima fronte difficilia videantur non semper ad Algebram recurrendum esse." MP, 5, p. 336.

[70] Problem 59 corresponds to Problem 55 of *Lucasian Lectures on Algebra*. The following two problems, 60 and 61, are generalizations, namely, "To describe a conick section which shall pass through four given points, and in one of those points shall touch a right line given in position" and "To describe a conick section which shall pass through three given points, and touch right lines given in position in two of those points." MP, 5, pp. 308–15.

$y^2 = 0$. The five conditions of the problem (i.e., the fact that the conic must pass through five given points) translate into a system of five equations, which, when resolved, determine the coefficients. This is the end point of the analysis: "And from its equation the conic section will—by Descartes' method—itself be determined."[71]

The determination, or construction, of the conic constitutes the synthesis and is achieved, Newton's suggested, by Descartes' method; by determining the parameters of the conic in function of the coefficients of its equation and by using the classic Apollonian construction via the section of a cone with a plane. In Newton's opinion, this was a very complicated procedure compared to the organic description he had communicated to Collins in 1672.

The text of the *Lectures* does not reveal the extent of Newton's opposition to this Cartesian solution. From the many manuscript pages considered here, it can be surmised that Newton deemed algebraic analysis unsatisfactory because it did not lead to synthesis through a simple reversal of steps.

5.5 Tensions

The section on the resolution of geometrical problems in *Lucasian Lectures on Algebra* ends with a comparison between algebraic and geometrical analyses. The whole purpose of this comparison was to "convey the point that in problems which at first glance seem difficult there is no need always to have recourse to algebra."[72] Such a comparison was the result of the extensive research on porisms that Newton had undertaken in the late 1670s, convinced as he was of the superiority of the ancient method.

Newton had serious, even passionate, reservations about the algebraic analysis of locus problems. This struck him as more cumbersome and less elegant than geometry. But the main criticism that he leveled at the algebraic analysis of locus problems proposed by Descartes was that it did not lead to synthesis in a simple way. The analysis of Problem 59 in terms of projective geometry was considered by Newton as preferable both in terms of economy and elegance, and because it led immediately to a well-grounded synthesis: to a construction of the conic based on acceptable postulates.

In sum, Newton criticized the lack of elegance and ease in Cartesian analysis of indeterminate problems and believed Cartesian synthesis was based on "spurious" postulates: point-wise constructions, motion of curves, construction with threads, and intersection of cone and plane (barren speculation). He favored expressing the contents of known and unknown segments that have a given ratio to one another,

[71] MP, 5, p. 317. "Et ex ea aequatione per methodum Cartesij determinabitur Conica sectio." MP, 5, p. 316.
[72] MP, 5, p. 337.

and using projective properties to compose the locus by organic descriptions based on acceptable postulates, namely, rotating rulers.

Nevertheless, Newton was a master in Cartesian algebra, the "common analysis of the moderns," and when he had to deal with a challenging problem, the classification of cubic curves, he made use of algebraic techniques that were at odds with his program of restoration of ancient analysis. Chapter 6 explores the tensions that this divide between practice and methodology engendered.

6 Beyond the Cartesian Canon: The Enumeration of Cubics

> . . . it's plain to me by ye fountain I draw it from, though I will not undertake to prove it to others.
>
> — Isaac Newton, 1676

> [Newton] has often spoken in the manner of prophets, who speak of that which one cannot see.
>
> — Jean S. Bailly, 1785

6.1 Studies on Cubics

6.1.1 Early Work

Newton's first attempts to enumerate cubic curves date from the late 1660s or early 1670s.[1] In these early works he was able to reduce, via a change of coordinate axes, the general form of a third-degree polynomial to four cases.[2] He reconsidered the classification of cubic curves in the late 1670s,[3] reaching most of the results that he later, in 1695, systematized in a slim treatise that was to appear in 1704 as an appendix to the *Opticks* under the title *Enumeratio Linearum Tertii Ordinis*.[4]

It is interesting first to consider the text of the printed *Enumeratio* and then step back to the extant manuscripts on cubics that were written in the 1670s and 1690s. The direction this chapter takes is from printed source to manuscript. A

Epigraph sources: (1) Newton to Collins (November 8, 1676). *Correspondence*, 2, p. 180. (2) "[Newton] avoit souvent parlé à la manière des Prophetes, qui disent ce qu'on ne peut voir." Bailly, *Histoire de l'Astronomie Moderne* (1785), 3, p. 150.

[1] There is disagreement between Whiteside and Westfall. Whiteside (MP, 2, p. 11) tentatively dates the early "Enumeratio Curvarum Trium Dimensionum" (Add. 3961.1, ff. 2r–3r, 10r–13r,, 6r–9r, 14r–16r, 22r–30r) in 1667-8, whereas Westfall indicates 1670 (*Never at Rest* (1980), p. 197). The critical edition can be found in MP, 2, pp. 10–89.

[2] The coordinate transformation (a translation plus a rotation of the axes) led Newton to a fantastic equation with 84 terms, which can be admired in MP, 2, p. 12.

[3] In a group of rather scattered annotations both in the Portsmouth and in the Macclesfield Collections that Whiteside edited in MP, 4, pp. 346–401. The dating is particularly uncertain.

[4] Newton dealt with the central projection of cubics in the first book of "Geometriae Libri Duo" (MP, 7, pp. 410–35). Preliminary notes to the *Enumeratio* are in MP, 7, pp. 579–87. The final manuscript of the *Enumeratio* as sent to the printer (MS Add. 3961.2, ff. 1r–14r) is in MP, 7, pp. 588–645, with variants in MP, 7, pp. 646–53. The *Enumeratio* first appeared in Newton, *Opticks* (1704), pp. 139–62 (+ 6 Tables).

similar perspective reveals an interesting aspect of Newton's mathematical work: his publication policy. The printed text of the *Enumeratio* presents a number of obscurities, since many demonstrations are omitted. Consequently, revealing the hidden subtext of the *Enumeratio* became a highly esteemed exercise in Newton's circle: an exercise in which James Stirling, Colin Maclaurin, and Patrick Murdoch all engaged.[5] The exegesis of Newton's enigmatic text was facilitated by enjoyment of the confidence of the "illustrious author" and by the possibility of perusing his manuscripts, privileges that were only accorded to a few lucky acolytes (§16.2). Now Whiteside's edition of *Mathematical Papers* makes possible a similar modern exercise. It is possible to argue on the basis of the extant manuscripts that most of the results of the *Enumeratio* were achieved via the employment of algebraic rather than geometrical analysis.

6.1.2 *Enumeratio Linearum Tertii Ordinis*

The *Enumeratio* can be divided into seven sections.[6]

In the first section Newton dealt with the definitions of order and genus. According to his terminology, conics are lines of second order and curves of first genus (he did not consider the straight line a curve). Cubics are thus lines of third order and curves of second genus. The order is equal to the degree of the defining algebraic equation. Newton also observed that lines of nth order could be cut in n points at the most by a straight line: a conic could be cut in two points at the most, a cubic in three points at the most, and so on. Newton also defined mechanical curves as "lines of infinitesimal order." He stated that similar curves—the spiral, the cycloid, the quadratrix—could be cut by a straight line in an infinite number of points.

In the second section Newton introduced the principal properties of cubics, such as those pertaining to diameters, vertices, centers, axes, and asymptotes. He basically extended some well-known definitions and properties of conic sections to curves of a higher order.

In the third section Newton reduced the general equation of a cubic to four canonical forms, namely,

$$xy^2 + ey = ax^3 + bx^2 + cx + d \tag{6.1}$$

$$xy = ax^3 + bx^2 + cx + d \tag{6.2}$$

$$y^2 = ax^3 + bx^2 + cx + d \tag{6.3}$$

$$y = ax^3 + bx^2 + cx + d. \tag{6.4}$$

[5] Stirling, *Lineae Tertii Ordinis Neutonianae* (1717); Maclaurin, *Geometria Organica* (1720); Murdoch, *Neutoni Genesis Curvarum per Umbras* (1746).

[6] The original version, published in 1704 in Newton, *Opticks* (1704), is not divided into sections. This useful subdivision appears in 1711 in William Jones's edition of Newton's mathematical tracts.

This by no means trivial result allowed Newton to divide cubics into four cases.[7]

Each case is subsequently subdivided into classes, genera, and species (figure 6.1).

In the fourth section Newton classified 72 different species of cubics in total. It is well known that six cubics are missing from Newton's printed *Enumeratio*: four were added by Stirling in 1717, one by François Nicole in 1731, and one by Nicolaus Bernoulli in 1733.[8]

Canonical form or Case			
I. $xy^2+cy = ax^3 + bx^2 + cx + d$	Redundant Hyperbolas (a positive)	adiametral	9
		monodiametral	12 +2
		tridiametral	2 +2
		[asymptotes concurrent]	9
	Defective Hyperbolas (a negative)	adiametral	6
		monodiametral	7
	Parabolic Hyperbolas ($a = 0$)	adiametral	7
		monodiametral	4 + 2
	Hyperbolism of Conics ($a = 0$, b=0)	hyperbolic	4
		elliptic	3
		parabolic	2
II. $xy = ax^3 + bx^2 + cx + d$	(The trident)		1
III. $y^2 = ax^3 + bx^2 + cx + d$	Diverging Parabolas		5
IV. $y = ax^3 + bx^2 + cx + d$	(The Cubic Parabola)		1

Figure 6.1

Newton's classification of nondegenerate cubics (completed with the six cubics missing from the *Enumeratio*). The degenerate forms of a conic and a straight line and of three straight lines are excluded. After: Rouse Ball, "On Newton's Classification of Cubic Curves" (1891), p. 114. By Compomat, s.r.l. ©Niccolò Guicciardini.

[7] In Newton's mathematical manuscripts it is possible to find different proofs of this statement. One is based on an algebraic change of coordinate system. The other is based on reasoning assisted by geometrical intuition. This second process surfaces in the printed text of the *Enumeratio*. Both these processes are analyzed in detail by Rouse Ball, "On Newton's Classification of Cubic Curves" (1891), pp. 107–113. See MP, 4, pp. 358–68.

[8] It is interesting to note that, as Whiteside discovered by analyzing a manuscript dating from the 1690s, Newton had identified all the six missing cubics. Why he did not take notice of these in the printed *Enumeratio* is unknown. See Whiteside's comments in MP, 7, p. 426 (n. 54) and p. 431 (n. 65). David Gregory, after being shown some mathematical manuscripts by Newton himself, noted: "sunt 16 genera Curvarum secundi generis, et 76 Curvae Newtonus conscripsit tractatum de illis quem mihi impertietur ut eum edam." *Correspondence*, 4, p. 277.

The second, third, and fourth cases give y as an explicit function of x. The first case is more difficult. However, if $e = 0$, then y is an explicit function of x; when $e \neq 0$, Newton studied the equation (6.1) in the form

$$\left(xy + \frac{1}{2}e\right)^2 = ax^4 + bx^3 + cx^2 + dx + \frac{1}{4}e^2. \tag{6.5}$$

In order to classify cubics, Newton first considered the value of the coefficients and then the number and position of roots of the right-hand sides of equations (6.2)–(6.5).

In the fifth section of the *Enumeratio*, Newton stated that "if onto an infinite plane lit by a point-source of light there should be projected the shadows of figures," then all cubic curves could be generated by projecting one of the five divergent parabolas (i.e., a curve belonging to one of the five species into which case 3 is subdivided), just as all conic sections can be obtained as projections of the circle.

In the sixth and seventh sections, which conclude the *Enumeratio*, Newton dealt with the organic description of curves (§5.4) and with the use of curves in the construction of the roots of equations (§4.3).

The novelty of Newton's approach to cubic curves compared to the ancient tradition of problem solving, to which he often referred as his supposed model, should not escape notice. In the Greek tradition curves are mainly seen as means of construction, not objects of study. It is clear that in the *Enumeratio*, Newton drew inspiration from Descartes, Jan de Witt, and especially Wallis.[9] Wallis had studied conics as loci of points satisfying second-degree algebraic equations in two unknowns; he had reduced second-degree polynomials to normal form and shown how the usual classification into hyperbola, parabola, and ellipse depends upon the coefficients. As far as third-degree curves, only a handful were known before Newton.[10] It was a natural step to examine the whole family of cubics. These, according to Descartes' canon, were the simplest curves after the conics. In his study of cubic curves, however, Newton found himself facing a wild variety of shapes: their complex graphs fascinated him but also reinforced his conviction that Descartes' criteria of simplicity were foreign to geometry.

[9] De Witt, *Elementa Curvarum Linearum*, in Descartes, *Geometria* (1659–61), pp. 153–340; Wallis, *De Sectionibus Conicis, Nova Methodo Expositis Tractatus* in *Operum Mathematicorum Pars Altera* (1656).

[10] The cissoid ($y^2(a-x) = x^3$), the Cartesian "parabola" ($axy = x^3 - 2ax^2 - a^2x + 2a^3$), the Cartesian folium ($x^3 - axy + y^3 = 0$), Wallis's cubic parabola ($y = ax^3$), and Neil's semicubic parabola ($y^2 = ax^3$), where $a \in \Re$.

6.2 Absence of Demonstrations in the *Enumeratio*

The *Enumeratio* provides a first illustration of some of the peculiarities in Newton's publication strategy that also emerge in the case of the *Principia* (see part IV). A tension is apparent between Newton's methodology, which gives pride of place to ancient geometry, and his mathematical practice, which is based on algebraic methods. Newton had developed most of his results on cubics by the early 1670s, just before the inception of his research on porisms and Apollonian geometry (§5.2); he systematized them in the 1690s, when his fascination with the ancient mathematicians and philosophers was at its peak. Newton's anti-Cartesianism notwithstanding, he achieved most of his results on cubics via application of Cartesian common analysis, and, I surmise, via the use of a newer analysis, namely, infinite series. In the mid 1670s a tension thus emerged between Newton's mathematical practice and the views on method that he developed when considering the use of analysis and synthesis in writings such as *Lucasian Lectures on Algebra* (§4.5, §4.6) and "Solutio Problematis Veterum de Loco Solido" (§5.4).

The tension between method and practice led Newton to structure his printed work on cubics, *Enumeratio Linearum Tertii Ordinis*, in such a way that the use of the common and new analyses is not made entirely explicit. While this notable characteristic of Newton's printed mathematical writings, their opacity, is well known, I believe it is little understood. In many cases Newton provided just a hint, or no trace at all, of a demonstration of his statements. Newton's way of presenting his results was often declaratory rather than argumentative; his readers often complained about this fact and tried to clarify obscure points by means of manuscript sources or oral communication. Yet even the study of Newton's mathematical manuscripts, a vantage point made possible by the publication of Whiteside's edition, is not always of great help. In many cases one can just conjecture how Newton might have obtained some of his deepest results.

The list of unproven mathematical statements in Newton's writings is long. For instance, his rule for enumerating imaginary roots (§4.1) was merely stated in *Lucasian Lectures on Algebra*. The *Principia* (see part IV) is a rich repertoire of such mysteries, which can be resolved in only a few cases by the study of extant manuscript sources. For the extraordinary statement about the projective classification of cubics as shadows of the five divergent parabolas (§6.4.1) Newton provided no proof whatsoever in the printed edition of the *Enumeratio*. Could it be that he only personally revealed it to his faithful acolytes? Newton's strategies of communication in mathematics require further clarification (see parts IV and VI).

One of the most disconcerting aspects of the *Enumeratio* is that it provides no proof of most of its propositions. Most notably, nowhere does one find proofs of the basic properties of diameters, asymptotes, and chords, stated in Section 2; or of the fact that every cubic can be generated by centrally projecting one of the five divergent parabolas, stated in Section 5. These are fundamental statements, astonishingly simple in their enunciation, yet they are not proven anywhere in the *Enumeratio*. Sections 3 and 4, devoted to a long classification of cubics, are perhaps less mysterious; here too, however, Newton limited himself to providing a classification and gave few clues about how such a classification might be achieved. Each cubic is carefully plotted in the beautiful figures that adorn the text: a world of strange objects, with ovals and branches extending to infinity (figure 6.2). But few instructions are given on how to plot the curves.

James Stirling commented on the *Enumeratio* in *Lineae Tertii Ordinis Neutonianae* (1717), where the algebraic character of Newton's work, most notably the

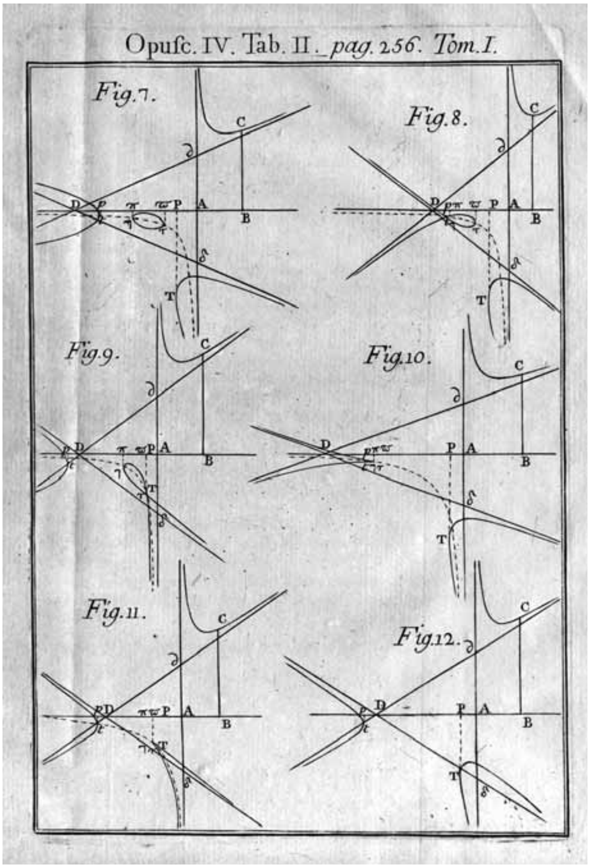


Figure 6.2

Newton's detailed drawing of cubic curves. When such diagrams appeared in the *Enumeratio* (1704), it was not easy to divine Newton's methods for plotting curves from equations in such fine detail. Source: Newton, *Opuscula Mathematica, Philosophica et Philologica* (1744), 1, Tab. II. Courtesy of the Biblioteca Angelo Mai (Bergamo).

use of infinite series, is spelled out (§6.5). Stirling's work to reveal the analysis behind Newton's published proofs was highly appreciated.

Generally speaking, several of Newton's mathematical printed works (most notably the *Enumeratio*, the *Principia*, and *De Quadratura*) are difficult to read because most of the proofs are either incomplete or completely lacking. This aspect of Newton's printed mathematical work is related to his conceptions concerning mathematical method. The case of the *Enumeratio*, from this point of view, is rather extreme, since in the work almost all traces of an algebraic demonstrative structure are concealed.

Such an elliptical style soon became the object of comments and complaints, which oscillated between frustration and reverence toward a man who flew so high yet was so immersed in his own thoughts that he did not even care to bow down to the level of common mortals who, after all, needed to be told how problems could be constructed and theorems proven.

Leibniz's anonymous review of William Jones's edition of the *Enumeratio*[11] in *Acta Eruditorum* contains one of the first critical reactions to the work:

> The illustrious Editor [William Jones] would have acquired a distinguished merit amongst the geometers, if he had, at least once, exhibited a demonstration of the number of lines of third order, which upon request Newton would not have denied: further indeed he would deserve merit if he were to publish it as an appendix or on another occasion.[12]

Similar complaints were not unusual. For instance, a typical critical evaluation of the *Enumeratio* can be found in the historical Preface to Vincenzo Riccati and Girolamo Saladini's *Institutiones Analyticae* (Bologna, 1765):

> Isaac Newton brought an enumeration of the lines of third degree to light, even though without any published demonstration, being the rules which have been employed just lightly touched upon, since he was more desirous to establish admiration for himself, rather than to instruct others.[13]

[11] Jones included the *Enumeratio* in his edition of Newton's mathematical tracts and excerpts of letters published in 1711. Newton, *Analysis per Quantitatum* (1711), pp. 67–92.

[12] "Sed egregie de Geometris meritus fuisset Cl. Editor, si demonstrationem numeri linearum tertii ordinis, quam petenti non denegaturus erat Newtonus, una exhibuisset: immo adhuc bene mereri poterit, si per modum appendicis aut alia occasione edat." [G.W. Leibniz], "Analysis per Quantitatum, Series, Fluxiones, ac Differentias" (1712), in part reproduced in MP, 2, pp. 259–62.

[13] "Isaacus Newtonus enumerationem linearum tertii gradus in lucem protulit, licet nulla edita demonstratione, regulisque quibus usus erat, minime attactis, quippe qui magis sibi ipsi admirationem comparare, quam alios edocere cupiebat." Riccati and Saladini, *Institutiones Analyticae* (1765–67), 1, p. x.

But Jean Paul de Gua de Malves wrote with veneration:

> This geometer, whose works are characterized by a unique sublimity, especially in this one seems to have elevated himself to an immense height, to which all other minds less penetrating and strong would have attempted in vain to attain. But the path he has followed in such a difficult enterprise escapes the sight of those who marvel at the degree of elevation to which he has arrived. The exception are a few light traces that he cared to leave in places which would have deserved that he would have stopped there for a much longer interval of time. These places, moreover, are almost always very far one from the other. If one desires to follow the same route, one is compelled to guide oneself along such distant intervals.[14]

Gabriel Cramer openly displayed a critical attitude, not devoid of moral reproach:

> It is deplorable that Newton was satisfied in displaying his discoveries without adding the demonstrations, and that he has preferred the pleasure of being admired to that of providing instructions.[15]

Talbot, who translated and commented upon the *Enumeratio* in the nineteenth century, came to Newton's defense. But although Talbot found "the criticisms of the French mathematicians [De Gua and Cramer] ill founded," he had to admit that "some explanation and illustration is wanted." Rouse Ball, who wrote about the *Enumeratio* in the 1890s, more bluntly observed that in Newton's treatise "no proofs of the propositions are given." Talbot went on to claim that the conciseness of Newton's treatise was due to the fact it had been hastily published in order "to vindicate the priority of his own [Newton's] discoveries." Rouse Ball refers to the "Advertisement" featured in the preface to the *Opticks*, where one can read that the author was concerned with "things being copied out" of a manuscript that "had been lent out."[16] However, in the "Advertisement," Newton was referring to his discoveries on the quadrature of curves: the manuscript tract that was circulating

---

[14] "Ce géomètre dont tous les ouvrages portent un caratctère singulier de sublimité, paroit en particulier dans celui-ci s'être élevé à une hauteur immense, à laquelle toute autre génie moins pénétrant et moins fort que le sien, auroit tenté vainemment d'atteindre: mais la route qu'il a tenue dans une enterprise si difficile, se dérobe aux yeux de ceux qui apperçoivent avec étonnement le degré d'élévation auquel il est parvenu. On doit en excepter quelques legères traces qu'il a eu soin de laisser sur son passage, aux endroits qui avoient mérité qu'il s'y arrétât plus long tems. Ces endroits, au reste, sont presque toujours assez distants les uns des autres. Si l'on se propose donc de suivre la même carrière, on est obligé se guider soi-même dans de long intervalles." Gua de Malves, *Usages de l'Analyse de Descartes* (1740), pp. xi–xii.

[15] "Il est facheux que M. Newton se soit contenté d'étaler ses découvertes sans y joindre les Démonstrations, et qu'il ait préféré le plaisir de se faire admirer à celui d'instruire." Cramer, *Introduction à l'Analyse des Lignes Courbes Algébriques* (1750), pp. viii–ix.

[16] Newton, *Opticks* (1704), Advertisement [n.p.]. See Talbot's preface in Newton, *Enumeration of Lines of the Third Order* (1861), p. vii, and Rouse Ball, "On Newton's Classification of Cubic Curves" (1891), p. 105.

was *De Quadratura*, not the *Enumeratio*. Newton's concern about priority was related to quadratures, not to the enumeration of cubics, a topic that aroused little interest during his own lifetime.

What one encounters here is the typical Newtonian attitude toward the publication of mathematical works. Most of Newton's mathematical works aroused astonishment for their results but criticisms for their lack of adequate fully spelled-out proofs. These reactions are evident even among Newton's contemporaries; they are a sign that Newton was following a strategy that seemed peculiar to the actors of his times. The idea that Newton's "conciseness" was due to haste or the necessity to secure priority is also often mentioned in the literature, especially after the nineteenth century.

In order to understand this aspect of Newton's publication policy, one must place the printing of his mathematical works in the context of the publication practices adopted in his time. One must also consider the role of other means of publication that Newton deployed, such as correspondence and the circulation of manuscripts. Similar considerations enable a focus on important aspects of Newton's ideas on mathematical method (see part VI).

Here the printed text of the *Enumeratio* is taken as a starting point to propose some conjectures about the mathematical proofs it implies.[17] It is most likely that Newton deployed both common Cartesian analysis and advanced algebraic techniques that go far beyond the methods of Descartes.

6.3 Common Analysis in the *Enumeratio*

6.3.1 A Puzzle in Section 2

In Section 2 of the *Enumeratio* the definitions of diameter, chord, center, axes, and so on, which since antiquity had been applied to conic sections, are extended to cubics. Newton also extended some of the properties valid for conics to cubic curves. It can be shown that all the properties of cubics that Newton listed in Section 2, which are rather difficult to visualize geometrically, can in fact be deduced from simple properties of the roots of their defining equations.[18]

Newton gave no indication of how a proof of these properties might be achieved. For instance, he wrote,

[17] Quotations are taken from Whiteside's translation of Add. 3961.2, ff. 1r–14r, the definitive version that was printed with minimal variations as an appendix to the *Opticks* in 1704. MP, 7, pp. 588–645.

[18] See the commentary by Antonio J. Durán Guardeño in Newton, *Análisis de Cantidades, Mediante Series, Fluxiones y Differencias. Con una Enumeración de las Líneas de Tercer Orden* (2003), pp. 148–56.

> *The ratio of the products contained under segments of parallels.* Just as in conics, when two parallels terminating on either side at the curve are cut by two parallels terminating on either side at the curve, the first by the third and the second by the fourth, the rectangle of the parts of the first is to that of the parts of the third as the rectangle of the parts of the second to that of the parts of the fourth; so when four such straight lines meet a curve of second kind [a cubic], each individually in three points, the parallelepiped of the parts of the first line will be to that of the parts of the third as the parallelepiped of the parts of the second line will be to that of the parts of the fourth.[19]

This is a generalization to cubics of a property of conics that Newton used in his solution of the Pappus problem (§5.3.3). The property is proven by Apollonius in the *Conics*, Book 3, Propositions 17–23. It can be formulated as follows.

> **The property of intersecting chords of a conic**:[20] If two parallel chords A_1A_2 and B_1B_2 of a conic are cut by two other parallel chords C_1C_2 and D_1D_2 meeting in O_1 and O_2 (see figure 6.3), the following equation holds:

$$\frac{O_1A_1 \times O_1A_2}{O_1C_1 \times O_1C_2} = \frac{O_2B_1 \times O_2B_2}{O_2D_1 \times O_2D_2}. \tag{6.6}$$

In the case of conic sections a geometrical proof is possible following Apollonius. In the case of a general cubic, however, no such geometrical proof is available for Newton.

6.3.2 Algebraic Generalizations

From the manuscript works on cubics that Newton composed in the 1670s it is possible to surmise that he adopted the algebraic procedure spelled out by Stirling in *Lineae Tertii Ordinis Neutonianae*.[21] First Newton must have considered an algebraic proof for conics, which he then generalized to cubic curves. Stirling's approach can be summarized as follows.

---

[19] MP, 7, p. 593.

[20] This is the property that Apollonius described in the opening lines of the *Conics* as essential to the construction of the Pappus problem. Another way of stating this property is by saying that if two chords A_1A_2 and C_1C_2 of a conic intersect in O_1, the ratio of the rectangles contained by their segments $(O_1A_1 \times O_1A_2)/(O_1C_1 \times O_1C_2)$ is proportional to the ratio of the squares on parallel diameters. Of course, this ratio does not change when the chords are moved parallel to one another. Therefore equation (6.6) holds.

[21] On pp. 78–9. The evidence for Newton's algebraic approach to this theorem is provided in manuscripts edited in MP, 2, pp. 90–104, and MP, 4, pp. 354–60. See esp. MP, 4, pp. 358–60, where Newton expressed the general equation of a cubic curve. See also the equivalent algebraic proof in Miller, "Newton, Aufzählung der Linien dritter Ordnung" (1953), p. 20, which I follow here.

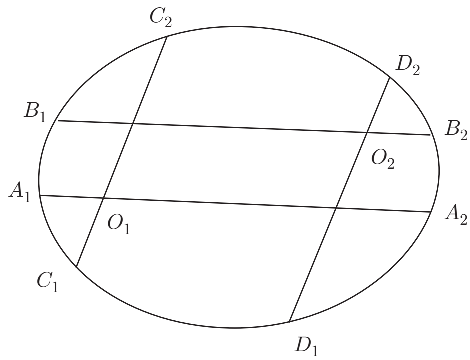


Figure 6.3
Diagram for Apollonius, III. 17–23. Source: By Compomat, s.r.l. ©Niccolò Guicciardini.

Figure 6.3 shows a conic, and two parallel chords cut by two other parallel chords. Now write the equation for a conic (with the typographic help of subindices, not employed by Stirling, and the notation $f(x, y)$ for a function, which is, in fact, an even more serious anachronism),

$$f(x, y) = a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00} = 0, \tag{6.7}$$

using a system of oblique coordinates with x-axis O_1A_2, and y-axis O_1C_2, so that O_1 is the origin.

For $y = 0$ Viète's property of the roots x_1 and x_2 of $f(x, 0) = 0$ yields

$$x_1 \cdot x_2 = O_1A_1 \times O_1A_2 = \frac{a_{00}}{a_{20}}, \tag{6.8}$$

and for $x = 0$ and the roots y_1 and y_2 of $f(0, y) = 0$,

$$y_1 \cdot y_2 = O_1C_1 \times O_1C_2 = \frac{a_{00}}{a_{02}}. \tag{6.9}$$

Now translate the coordinate system so that the origin becomes O_2 (that is, after a coordinate transformation $x = \xi + \alpha$ and $y = \eta + \beta$) the equation is changed so that the absolute term becomes $f(\alpha, \beta)$.[22] For $\eta = 0$ Viète's property of the roots

[22] In the new coordinate system the equation is

$$\begin{aligned} f(\xi, \eta) = a_{20}\xi^2 + a_{11}\xi\eta + a_{02}\eta^2 &\quad + \\ (2a_{20}\alpha + a_{11}\beta + a_{10})\xi &\quad + \quad (2a_{02}\beta + a_{11}\alpha + a_{01})\eta + \\ a_{20}\alpha^2 + a_{11}\alpha\beta + a_{02}\beta^2 + a_{10}\alpha + a_{01}\beta + a_{00} &\quad = \quad 0. \end{aligned}$$

ξ_1 and ξ_2 of $f(\xi, 0) = 0$ yields

$$\xi_1 \cdot \xi_2 = O_2B_1 \times O_2B_2 = \frac{f(\alpha, \beta)}{a_{20}}, \tag{6.10}$$

and for $\xi = 0$ Viète's property of the roots η_1 and η_2 of $f(0, \eta) = 0$ yields

$$\eta_1 \cdot \eta_2 = O_2D_1 \times O_2D_2 = \frac{f(\alpha, \beta)}{a_{02}}. \tag{6.11}$$

This completes the algebraic proof of Apollonius's theorem (equation 6.6).

An algebraic generalization to n-degree curves is immediately available. Let the equation be

$$f(x, y) = a_{n0}x^n \ldots + a_{0n}y^n \ldots + a_{00} \tag{6.12}$$

Assume that two parallel chords cut the curve in n points, A_1, A_2, ... , A_n, and B_1, B_2, ... , B_n, respectively. Two other parallel chords cut the curve in C_1, C_2, ... , C_n, and D_1, D_2, ... , D_n. Further, the first chord meets the third in O_1, and the second meets the fourth in O_2.

Following the same steps, Viète's property applied to the roots of $f(x, 0)$ and $f(0, y)$ yields

$$x_1 \cdot x_2 \ldots x_n = O_1A_1 \times O_1A_2 \ldots O_1A_n = (-1)^n \frac{a_{00}}{a_{n0}} \tag{6.13}$$

and

$$y_1 \cdot y_2 \ldots y_n = O_1C_1 \times O_1C_2 \ldots O_1C_n = (-1)^n \frac{a_{00}}{a_{0n}}. \tag{6.14}$$

After the transformation of coordinates, $x = \xi + \alpha$ and $y = \eta + \beta$, one obtains

$$\xi_1 \cdot \xi_2 \ldots \xi_n = O_2B_1 \times O_2B_2 \ldots O_2B_n = (-1)^n \frac{f(\alpha, \beta)}{a_{n0}} \tag{6.15}$$

and

$$\eta_1 \cdot \eta_2 \ldots \eta_n = O_2D_1 \times O_2D_2 \ldots O_2D_n = \frac{f(\alpha, \beta)}{a_{0n}}. \tag{6.16}$$

Thus,

$$\frac{O_1A_1 \times O_1A_2 \ldots O_1A_n}{O_1C_1 \times O_1C_2 \ldots O_1C_n} = \frac{O_2B_1 \times O_2B_2 \ldots O_2B_n}{O_2D_1 \times O_2D_2 \ldots O_2D_n}. \tag{6.17}$$

The property of cubic chords enunciated by Newton,

$$\frac{O_1A_1 \times O_1A_2 \times O_1A_3}{O_1C_1 \times O_1C_2 \times O_1C_3} = \frac{O_2B_1 \times O_2B_2 \times O_2B_3}{O_2D_1 \times O_2D_2 \times O_2D_3}, \tag{6.18}$$

is a particular case of equation (6.17).

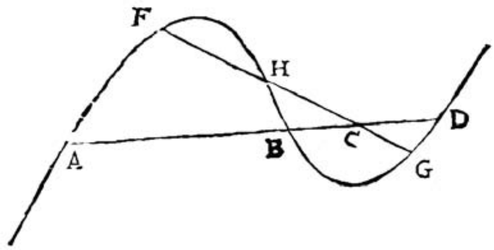


Figure 6.4
Diagram for Stirling's proof of equation (6.18). Two chords cut a cubic curve in F, H, G and A, B, D, respectively. The chords meet in C. Stirling proved that the ratio $(AC \times BC \times DC)/(FC \times HC \times GC)$ does not change when the chords are translated without changing their direction. Source: Stirling, *Lineae Tertii Ordinis Neutonianae* (1717), p. 78. Courtesy of the Biblioteca Angelo Mai (Bergamo).

In the *Enumeratio*, Newton never hinted at any proof of this generalization; he simply provided a geometrical property valid for the intersecting parallel chords of cubics. It is clear, however, that Newton must have followed a generalization of the preceding algebraic proof to third-degree polynomials. As Stirling remarked after providing the algebraic proof of equation (6.18) (figure 6.4),

> It suffices here incidentally to note that, proceeding with this universal method, that is by reasoning on the basis of equations, are manifest not only the properties of conic sections—to which the ancients arrived with so much labor, and which could be demonstrated with so much obscurities, by a method which cannot be extended to other curves—but also the properties of curves of superior order.[23]

Stirling was praising the advantages offered by modern algebra, something implied but never explicitly stated in the *Enumeratio*.

6.4 Projective Geometry in the *Enumeratio*

6.4.1 A Puzzle in Section 5

Section 5, devoted to the "Genesis of Curves by Shadows," is so concise that it can be quoted in full:

[23] "Sufficiat hic obiter annotare, quod hâc methodo universali procedendo, scilicet argumentando a naturis aequationum, patescunt non solum sectionum Coni proprietates, quas tanto labore adinvenerunt veteres, & tot ambagibus demonstratas dederunt, idque methodo quae ad alias curvas extendi nequit; sed & proprietates curvarum omnium ordinum superiorum." Stirling, *Lineae Tertii Ordinis Neutonianae* (1717), p. 79.

> If onto an infinite plane lit by a point-source of light [a puncto lucido] there should be projected the shadows of figures, the shadows of conics will always be conics, those of curves of second kind [cubics] will always be curves of second kind, those of curves of third kind will always be curves of third kind, and so on without end.
>
> And just as the circle by projecting its shadow generates all conics, so the five divergent parabolas [figure 6.5] by their shadows generate and exhibit all other curves of second kind; while in this manner certain simpler curves of other kinds can be found which by their shadows cast by a point-source of light onto a plane shall delineate all other curves of the same kinds.[24]

This is all Newton had to say about the genesis of curves by shadows.

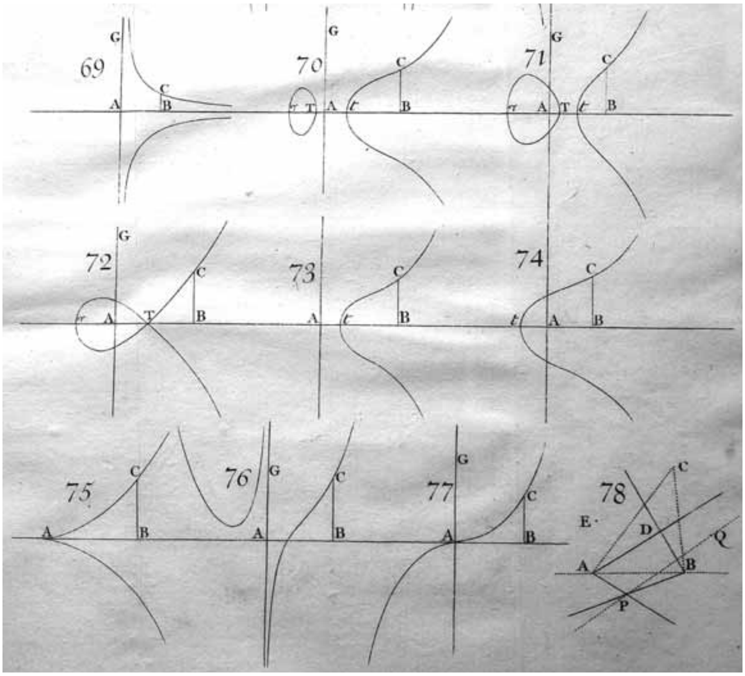


Figure 6.5

The five divergent parabolas. Equation $y^2 = ax^3 + bx^2 + cx + d$. If the three real roots of the right-hand-side third-degree polynomial are equal, one has the semicubic parabola (e.g., $y^2 = x^3$; number 75); three real unequal roots, a bell with an oval (e.g., $y^2 = (x+1)(x+2)(x-1)$; numbers 70 and 71); two of three real roots equal, parabola nodated by touching an oval (e.g., $y^2 = (x-1)^2(x+1)$; number 72) or punctuated by having the oval infinitely small (e.g., $y^2 = x^2(x-1)$; number 73); two imaginary roots and one real root, bell-like (e.g., $y^2 = (x+1)(x^2+4)$; number 74). Source: Newton, *Opticks* (1704), Tab. VI. Courtesy of the Biblioteca Angelo Mai (Bergamo).

[24] MP, 7, p. 635.

How and when did Newton achieve this profound result? The question was certainly in the readers's mind. Before the publication of Newton's *Mathematical Papers* it was unclear how an answer might be found. Even Whiteside admitted this:

> How and when Newton first achieved this insight that the numerous species of cubic curve fall into one or other of five projectively distinct "kinds" his extant papers do not reveal. We would guess that it came to him as an unexpected bonus while mulling over the series of sketches of the main individual species of cubic which he had prepared long before for his *Enumeratio*.[25]

Two nineteenth-century commentators, Talbot and Rouse Ball, provide different explanations. Rouse Ball believed that Newton relied on a transmutation of curves of the "same analytical order one into another," as expressed in Lemma 22, Book 1, of the *Principia*:

> I have little doubt that Newton had arrived at this remarkable result, which proved a puzzle to most of his contemporaries, by the method of projection indicated in the *Principia*, Bk. I, sect. 5, lemma xxii.[26]

Talbot, by contrast, surmised that Newton might have followed a geometrical procedure. An inspection of the extant manuscript does not allow a final answer. It seems that both algebra and geometry contributed to Newton's understanding.[27]

6.4.2 Rouse Ball's Interpretation

I first consider Rouse Ball's interpretation by turning to Lemma 22.

Right from the start, it must be stated that the transformation of curves described in Lemma 22 does not occur in any of Newton's mathematical manuscripts written before the *Principia*. Following Rouse Ball, then, one can only speculate

---

[25] MP, 7, p. 413, n. 27.

[26] Rouse Ball, "On Newton's Classification of Cubic Curves" (1891), p. 123. Note that Whiteside disagrees with Rouse Ball: "[W]e can place little faith in Rouse Ball's confident surmise." MP, 7, p. 414, n. 30.

[27] It is highly unlikely that Newton first achieved his result by algebraic means. In this case he should have had an algebraic expression for the section of a cone, which has one of the five divergent parabolas as base, and a plane. Newton dealt with the easier problem of algebraically determining the section of a circular cone with a plane in *Lucasian Lectures on Algebra* (= Problem 32 of *Arithmetica Universalis*). See MP, 5, p. 214. This is basically what François Nicole and Alexis Claude Clairaut did in 1731. See Talbot in Newton, *Enumeration of Lines of the Third Order* (1861), p. 83, and MP, 7, p. 414, n. 30. Clairaut's result was communicated to the French *Académie des Sciences* a few days later than a paper by Nicole on the same subject. A demonstration given by Patrick Murdoch in *Neutoni Genesis Curvarum per Umbras* (1746) is instead similar to the geometrical one elaborated by Newton in a long manuscript on geometry, which was not, so it seems, available to his contemporaries with the exception of a few acolytes; the manuscript is reproduced in MP, 7 (the pages pertinent to the generation of cubics by central projection of the five divergent parabolas are pp. 410–33).

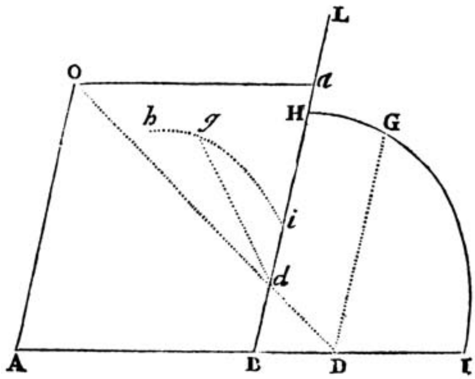


Figure 6.6
Diagram for Lemma 22, Book 1, of the *Principia*. Source: Newton, *Philosophiae Naturalis Principia Mathematica* (1726), p. 87. Courtesy of the Biblioteca Angelo Mai (Bergamo).

whether Newton had something similar to Lemma 22 in mind before writing the *Principia*.[28]

This lemma teaches how "To change figures into other figures of the same class." The figure to be transmuted is curve HG (figure 6.6). Draw the parallel straight lines AO and BL cutting any given third line AB in A and B, respectively. Then from some point O in the line AO draw the straight line OD. Let d be the point of intersection between OD and BL. From the point d erect the ordinate dg (one can choose any angle between the new ordinate dg and the new abscissa ad). The new ordinate and abscissa must satisfy the following conditions:

$$AD = \frac{AO \times AB}{ad} \tag{6.19}$$

$$DG = \frac{AO \times dg}{ad}. \tag{6.20}$$

Now suppose that point G "be running through all the points in the first figure [HGI] with a continual motion; then point g—also with a continual motion—will run through all the points in the new figure [hgi]."

Newton was clearly trying to conceive the most general transmutation between figures that preserved the order. Indeed, if one has an "equation which gives the

---

[28] Whiteside cautiously surmised (MP, 6, p. 271) that Newton might have found inspiration in the method of plani-coniques appended in La Hire, *Nouvelle Methode en Geometrie pour les Sections des Superficies Coniques et Cylindriques* (1673), pp. 75–94. According to Whiteside, the book was bought soon after its publication by the Cambridge University Library, reviewed in *Philosophical Transactions* for March 1676, and referred to by Hooke in his letter to Newton dated November 24, 1679 (*Correspondence*, 2, p. 298).

relation between the abscissa AD and the ordinate DG," the transformed equation relating the new abscissa ad and the new ordinate dg will be of the same degree.[29] Conics will be transformed into conics, cubics into cubics, and so on. This is an important point: the degree is invariant under central projective transformation; that is why the classification of curves by degree is meaningful and well-founded for Newton. It is an algebraic classification that captures an important geometrical invariant property, one of those properties that Newton speculated the ancients might have studied in their heuristic researches on the hidden field of geometrical analysis (§5.2).

This last points clearly emerges in what Newton said about projective classification of lines in the manuscript "Geometriae Libri Duo," dating from the 1690s:

> And hence, according to the number of points in which any line can be cut by a straight line, there arises the distinction of lines into degrees [or orders[30]]. ... If some line be looked at through a translucent plane by an eye situated outside its plane, and in that plane its apparent place or ... projection be marked, the projected line will be of the same order as the projecting one. ... In this way the ancients derived from the circle all figures of the second order and thence named them conic sections.[31]

Note that any point in the figure HGI that belongs to the line AO [the horizon line] will be sent to infinity because of equation (6.20). Moreover, the projection of a tangent or an asymptote of HGI is always either a tangent or an asymptote of the new figure hgi.

It is appropriate here to recall Newton's definition of horizon line (figure 6.7):

> **Definition of horizon line**. Let us name the "horizon" that plane which passes through a point-source of light and is parallel to the plane of the projected line, and the "horizon line" that line in which the horizon cuts the plane of the projecting line.[32]

We note here, in passing (we shall get back to this point when dealing with fluxions in §8.3), that Newton defined the transformation in kinematic terms: the projected curve hgi is generated by a motion of point g that is regulated by the motion of point G along HGI.

---

29 *Principles*, p. 494.

30 Here Newton conflates the terms degree and order.

31 MP, 7, p. 411. At times, Newton conceives of central projection as an eye looking through a translucent plane; at other times, as a shadow projected by a point source of light. For more on the "Geometriae Libri Duo," see part V. See also *Lucasian Lectures on Algebra*: "In contemplating curves and deriving their properties I commend their [the mathematicians of more recent times] distinction into classes in line with the dimensions of the equations by which they are defined." MP, 5, p. 425.

32 Adapted from MP, 7, p. 417. The term vanishing line is also employed.

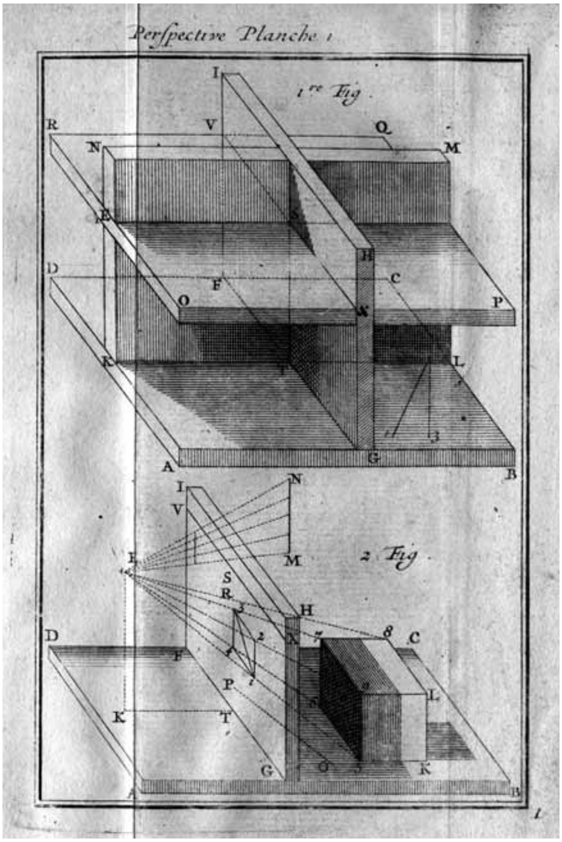


Figure 6.7
Central projection. The line VX is the horizon line (*ligne horizontale*), often called the vanishing line. Points belonging to the horizon line are projected at an infinite distance by a light-source placed at E. Source: Ozanam, *La Perspective Theorique et Pratique* (1711), Planche I. Courtesy of the Biblioteca Comunale dell'Archiginnasio (Bologna).

In the *Principia*, Newton stated that this lemma is "useful for solving more difficult problems by transmuting the figures into simpler ones." Indeed, any converging lines are transformed into parallels by positioning the horizon line AO in such a way that it passes through the intersection S of the converging lines (S is sent to infinity). Further, this lemma is useful for solving solid problems, that is, problems whose construction is given by the intersection of conic sections. For instance, a straight line and a conic can be "turned into a straight line and a circle."[33]

[33] *Principles*, pp. 494–5.

It is Rouse Ball's guess that Newton was assisted by algebraic transformations (equations (6.19) and (6.20)) in proving that all the cubics can be generated by projecting the five divergent parabolas.[34]

6.4.3 Talbot's Interpretation

It might be contended, *contra* Rouse Ball, that Newton, while writing on cubics in the 1670s—i.e., before the *Principia*—was guided by geometrical intuition rather than by algebra. This is Talbot's opinion. It is interesting to note that Patrick Murdoch in his commentary on the *Enumeratio*, published in 1746, adopted a similar geometrical approach.[35]

In support of Talbot's interpretation one can refer to a manuscript written by Newton in the mid-1690s. In an interpolation of Book 1 of "Geometriae Libri Duo" (§14.1), Newton generated all the cubic curves (even the famous six cubics missing from the printed *Enumeratio*) by projection of the five divergent parabolas, following a procedure very close to the one divined by Talbot.[36] This was all rather difficult to discern, however, for readers of the printed *Enumeratio*, and it may be that only a few lucky ones were allowed to see the manuscript during Newton's lifetime.

Newton began by noting that the position of the horizon line determines the nature of the asymptotes of the projected line (or curve, as we would say):

> [E]very projecting line will yield as many species of projections as there are cases of position of the horizon line. Should the horizon line cut the projecting line somewhere, that intersection will generate in the projection two legs of hyperbolic kind stretching round the same asymptote in opposite direction to infinity, this on the same side of the asymptote if the intersection be a point of inflection, but otherwise on opposite sides, while the asymptote will be the projection of the straight line

[34] Of course, the transformations of coordinates (6.19) and (6.20) are exactly those occurring between figures centrally projected from one plane to another. As Turnbull observed, Newton's formulas are easily rewritten in algebraic notation on taking $AD = x$ and $DG = y$, referred to AB and AO as axes, and $ad = x'$ and $dg = y'$, referred to aB and the parallel to dg through a as new axes. One gets $x = m/x'$ and $y = ly'/x'$ (l and m constants). Turnbull, *The Mathematical Discoveries of Newton* (1945), pp. 55–6. These algebraic transformations should have allowed Newton to verify how some of the simplest cubic curves can be obtained by projection of one of the divergent parabolas. For instance, it is easy to show how the semicubic parabola can be projected into the cubic parabola. Miller, "Newton, Aufzählung der Linien dritter Ordnung" (1953), p. 29; Talbot in Newton, *Enumeration of Lines of the Third Order* (1861), p. 82.

[35] Murdoch, *Neutoni Genesis Curvarum per Umbras* (1746), pp. 74–126. Whiteside noted small lacunae in Talbot's enumeration. Talbot's procedure is, however, perfectly cogent. See MP, 7, p. 420, n. 37.

[36] See MP, 7, pp. 411–35. Talbot and possibly Rouse Ball could base their conjectures on a manuscript (Add. 3961.3, ff. 17r–18r; see MP, 7, p. 418) that provides only a sketch of the much more detailed treatment of cubics found in MP, 7, pp. 411–35.

> touching the projecting curve at the point of intersection; and there will be as many pairs of legs of this sort in the projection as there are intersection of the horizon line with the projecting curve.[37]

Newton considered other positions of the horizon line, which can either "touch" the projecting curve or be an asymptote of the projecting curve.

The basic fact when considering the central projection of a curve on a plane of projection is that all points of the curve will be projected on the plane of projection, except for those that belong to the horizon line AO; these points are projected to an infinite distance.[38]

As in the case considered in §6.3, it is useful to start with conic sections and attempt a generalization to cubics. In "Geometriae Libri Duo," Newton began his treatment of cubics by making the elementary observation that when the projecting line is a circle, the horizon line will either cut the circle in two points and the projected line will be a hyperbola, or in one point and the projected line will be a parabola; instead, if the horizon line is outside the circle, the projected line will be an ellipse.[39]

Extending this procedure to cubics, as Newton did in the subsequent folios, is a rather difficult exercise of geometrical intuition. So, while the algebraic generalization of results proven for conics to cubics is straightforward (§6.3), generalizing geometrical reasoning from conics to cubics proves extremely difficult.

Consider one particular case: a divergent parabola whose graph has a bell with an oval. The horizon line can be either parallel or oblique to the ordinates. When it is parallel one must distinguish eleven different positions of the horizon line: each of these positions generates a different cubic curve by central projection (figure 6.8). For each of the divergent parabolas one has to consider the projections obtained by placing the horizon line in different positions, parallel and oblique to the ordinates.[40]

Newton ultimately showed that as the projection of the circle generates all conic sections, so the projections of the five divergent parabolas generate all the 72+6 cubics. The treatment of conic sections is instead much simpler. Only one projecting curve (the circle) has to be considered, and only three positions of the horizon line are to be studied.

It is at this juncture that I would like to advance a hypothesis concerning Newton's procedures in projective geometry. As previously noted, Newton's algebraic insights into projective invariant properties might have been aided by geometrical

---

[37] MP, 7, p. 417. I slightly alter Whiteside's translation by rendering "linea horizontalis" as horizon line rather than horizontal, and "crus" as leg rather than branch.
[38] Newton, *Enumeration of Lines of the Third Order* (1861), p. 76.
[39] MP, 7, p. 419.
[40] One can can verify, for instance, that "if the horizontal line passes through the cusped vertex of a [semicubical] parabola [$y^2 = ax^3$] and this at an angle of contact, the projection will be a Wallisian parabola [$y = ax^3$]." MP, 7, p. 421.

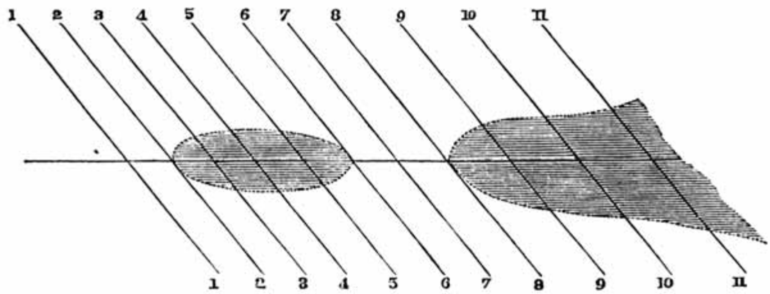


Figure 6.8
Talbot's study of the central projections of the divergent parabola ("bell with an oval"; see figure 6.5, numbers 70 and 71) when the horizon line is parallel to the ordinates. Talbot considered 11 positions of the horizon line and showed how 11 different cubics could be generated by central projection. After: Newton, *Enumeration of Lines of the Third Order* (1861), p. 77. By Compomat, s.r.l. ©Niccolò Guicciardini.

intuition. One might surmise that Newton was also aided by some sort of real apparatus that enabled him to actually project the divergent parabolas, perhaps by means of a thin paper sheet and a point-like light source. The point-source of light he referred to in Section 5 (§6.4.1) might not have been metaphorical but rather a real light source. I have already surmised that in the case of organic curve descriptions Newton might have used real instruments (§5.4.3). Newton's mathematical tool kit probably included real devices that allowed simulations to take the place of abstract mathematical proofs. This conjecture might account for the assertive style and lack of demonstrations that one encounters in certain sectors of Newton's mathematical *Nachlass*, most notably in his work on organic constructions of higher-order curves and projective classification of cubics.

6.4.4 Newton's Interpretation

The lesson that Newton learned from his projective classification of cubic curves is again at odds with Descartes' defense of algebra as problematic analysis. First, contrary to what Descartes had stated, the curves defined by third-degree algebraic equations are far from simple: they show bewilderingly complex shapes. Second, Newton cultivated projective geometry as a means for attacking geometrical problems from a point of view alternative to Cartesian algebra. Most notably, it is the projective invariance of degree and tangency that allowed him to simplify problems by transmuting curves one into another. It is likely that Newton conceived of projective geometry as related to porisms, that is, one of the basic ingredients of ancient geometrical analysis that he polemically compared with Cartesian analysis.

Finally, when considering points projected at infinity, Newton was moving beyond the mathematical finitism that he believed Cartesian methodology implied.

6.5 New Analysis in the *Enumeratio*

6.5.1 A Puzzle in Section 4

How could Newton plot cubic curves? On this issue, too, the text of the *Enumeratio* is somewhat mysterious. In Section 4, Newton classified cubics and plotted their graphs; he also related the graphs to the coefficients and roots of polynomials in x and y. However, the reader is not informed of the methods that Newton deployed in order to study the shapes of cubics in such fine detail. Newton's manuscripts, however, provide some clues: his techniques, it is possible to surmise, were largely algebraic. I believe Newton had to rely on techniques that included advanced algebraic tools not contemplated in Descartes' canon, that is, the use of infinite series.

When Stirling published his commentary on the *Enumeratio* in 1717, he deployed infinite series in order to provide the demonstrations lacking from Newton's work. In order to understand Newton's printed text, one must often turn either to his manuscripts or to the works of his close followers, who rendered public what the master had left concealed. Of course, it is not certain that Stirling's methods coincide with those employed by Newton.

Commenting on Newton's classification of cubic curves in any detail would require too much space. The following sections first examine a cubic curve (parabolic hyperbola) by applying an approximation technique that Newton most probably deployed, and a cubic (redundant hyperbola), treated at the end of Section 3 of the *Enumeratio*, where Newton began his classification. For this second example, Newton most probably employed infinite series expansions.

6.5.2 The Parabolic Hyperbola

A technique that Newton might have used in the *Enumeratio* consists of making recourse to an algorithm often called the analytical parallelogram. This algorithm allows one to obtain qualitative information on the behavior of the curve near the origin or at infinity by discarding from its defining equation those terms that are negligible for small or large values of x and y. Newton's parallelogram is discussed further in chapter 7 because its main use was to allow infinite series expansion.[41]

[41] This method is described by Talbot in Newton, *Enumeration of Lines of the Third Order* (1861), pp. 88ff, and in Frost, *An Elementary Treatise on Curve Tracing* (2004), pp. 117–32. I use notations for a function in two variables $f(x, y)$ and for a polynomial $p(x)$ that were not employed by Newton.

A simple application of the analytical parallelogram to the parabolic hyperbola is the following (figure 6.9):

$$f(x, y) = x^2y + 3y^2 - 9x = 0. \tag{6.21}$$

One easily verifies that the graph touches the axes only at the origin (where $f(0, 0) = 0$) and that the equation is not defined for $y > \sqrt[3]{27/4}$ and for $x \in (-\sqrt[3]{108}, 0)$.[42]

For large x and small y, equation (6.21) is approximated by

$$x^2y - 9x = 0, \tag{6.22}$$

which gives two hyperbolic legs. For x and y large, equation (6.21) is approximated by

$$x^2y + 3y^2 = 0, \tag{6.23}$$

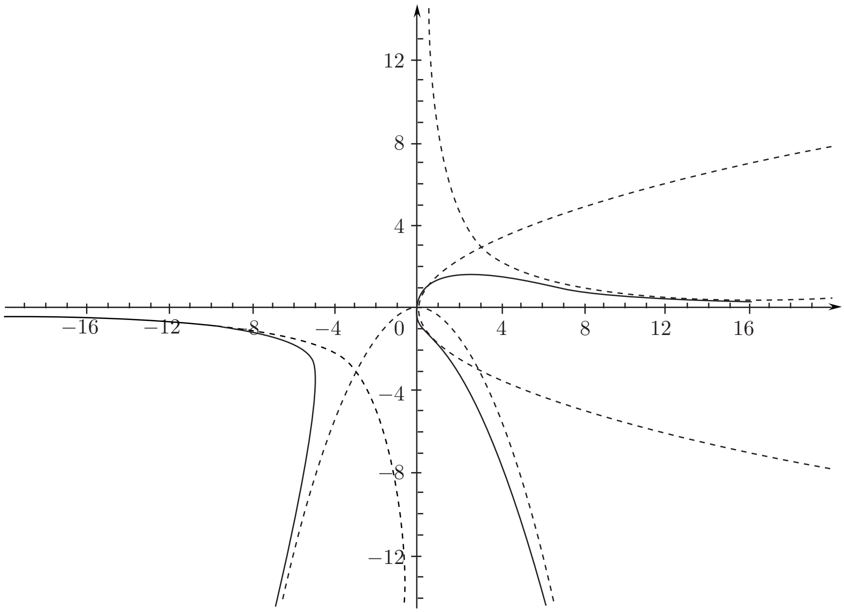


Figure 6.9
Graph of a parabolic hyperbola with equation $x^2y+3y^2-9x = 0$ (solid line). Approximated (dotted lines) for x and y large by $x^2y+3y^2 = 0$, for x large and y small by $x^2y - 9x = 0$, near the origin by $3y^2 - 9x = 0$. ©Niccolò Guicciardini.

[42] Indeed, solving for x, the discriminant is $81 - 12y^3$, and solving for y, the discriminant is $x^4 + 108x$.

which indicates two parabolic legs in the direction of the negative ordinates. At the origin the equation is approximated by

$$3y^2 - 9x = 0. \tag{6.24}$$

Newton's parallelogram offers a simple means of ascertaining which terms in an algebraic equation can be neglected in order to approximate the graph at the origin or at infinity.

6.5.3 The Redundant Hyperbola

An example of the redundant hyperbola has equation

$$xy^2 - ey = ax^3 + bx^2 + cx + d, \tag{6.25}$$

where, as usual, x is the abscissa AB, and y the ordinate BC (figure 6.10). This is the first cubic considered in the *Enumeratio*.[43] Without providing any trace of demonstration, Newton wrote,

> In the first case, if the term ax^3 is positive the figure will be a triple hyperbola with six hyperbolic legs which proceed to infinity in line with three asymptotes (none of which are parallel), a pair along one each in opposite directions. And if the term bx^2 is not lacking these asymptotes will mutually intersect one another in three points, so containing between themselves a triangle ($Dd\delta$) ... take $AD = -b/(2a)$ and $Ad = A\delta = b/(2\sqrt{a})$, and join Dd, $D\delta$, and Ad, and then Dd, $D\delta$ will be the three asymptotes.[44]

It was Stirling who provided a demonstration of this statement, which he promoted to the status of a Proposition 16 in his *Lineae Tertii Ordinis Neutonianae* (1717). By solving equation (6.25) for y, Stirling obtained:

$$y = \frac{e}{2x} \pm \sqrt{ax^2 + bx + c + \frac{d}{x} + \frac{e^2}{4x^2}} = \frac{\frac{1}{2}e \pm \sqrt{ax^4 + bx^3 + cx^2 + dx + e^2/4}}{x}. \tag{6.26}$$

Equation (6.26) is also featured in Newton's early studies on the redundant hyperbolas a clear sign that Stirling read through the text of the *Enumeratio* with much

[43] Note the minus sign in the left term: this is because Newton assumed that all the coefficients a, b, c, etc., are positive: "designant quantitates datas signis suis + et − affectas." Newton, *Opticks* (1704), p. 142. Today we would say that in equation (6.1), $e < 0$.

[44] MP, 7, p. 599. "In casu primo si terminus ax^3 affirmativus est Figura erit Hyperbola triplex cum sex cruribus hyperbolicis quae juxta tres Asymptotos (quarum nullae sunt parallelae) in infinitum progrediuntur, binae juxta unamquamque in plagas contrarias. Et hae Asymptoti si terminus bxx non deest se mutuo secabunt in tribus punctis triangulum ($Dd\delta$) inter se continentes ... cape $AD = -b/(2a)$ et $Ad = A\delta = b/(2\sqrt{a})$, ac junge Dd, $D\delta$, et erunt Ad, Dd, $D\delta$ tres Asymptoti." Newton, *Opticks*, p. 145 = MP, 7:, p. 598. I have slightly altered Whiteside's translation. Most notably, I render "crus" as leg rather than as branch.

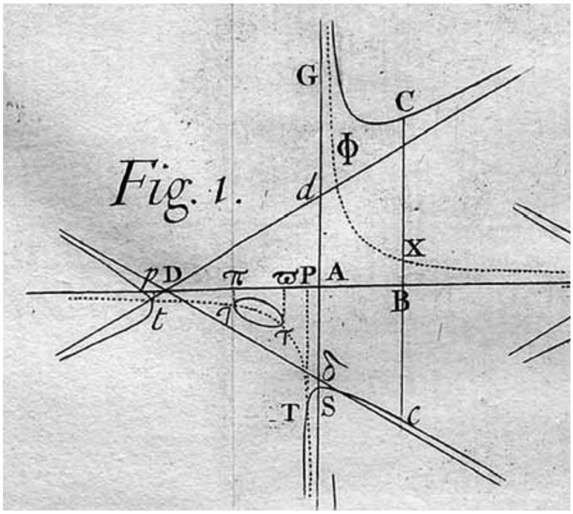


Figure 6.10
Graph of a redundant hyperbola from Newton's *Enumeratio*. The equation of the curve, (6.25), is $xy^2 - ey = ax^3 + bx^2 + cx + d$. This is the first cubic in Newton's classification. The position of the asymptotes and legs is determined in Proposition 16 of Stirling's *Lineae Tertii Ordinis Neutonianae* (1717). In the case of this cubic there are several assumptions to be taken into account concerning the coefficients of equation (6.25) (e.g., the fact that a is positive and $b \neq 0$). Newton further assumed that the roots of $p(x) = ax^4 + bx^3 + cx^2 + dx + e^2/4$ (which is under the radical in equation (6.26)) are all "real, unequal and of the same sign." One example might be $xy^2 - \sqrt{4 \cdot 24}y = x^3 + 10x^2 + 35x + 50$, which is obtained when $p(x) = (x+1)(x+2)(x+3)(x+4) = x^4 + 10x^3 + 35x^2 + 50x + 24$ (see figure 6.11). Let the four roots be represented by the segments AP, $A\omega$, $A\pi$, and Ap. The axis of the abscissae can be divided into intervals where the curve is not defined, since $p(x)$ is negative and therefore y is imaginary. It turns out that the curve is not defined in the intervals $P\omega$ and πp. In Corollary 2, Proposition 16, Stirling noted that in correspondence of the interval $\omega\pi$ there must be an oval contained in the triangle $Dd\delta$ because for any other shape it would be possible to cut the curve with a straight line in four points (which is, of course, impossible in case of a third-degree curve). Both Newton and Stirling based similar reasoning on intuitive assumptions about the meaning of terms such as oval and the fact that the curve must be smooth. Source: Newton, *Opticks* (1704), Tab. I. Courtesy of the Biblioteca Angelo Mai (Bergamo).

insight.[45] By applying Newton's theorem (namely, the binomial theorem; see §7.3), Stirling expanded the square root into a power series that converges for x "small."[46]

[45] See in "Enumeratio Curvarum Trium Dimensionum" the equation occurring at MP, 2, p. 18.

[46] "reducatur (per Theor. Neutoni) pars irrationalis in seriem eo citius convergentem, quo minor est x." Stirling, *Lineae Tertii Ordinis Neutonianae* (1717), p. 87.

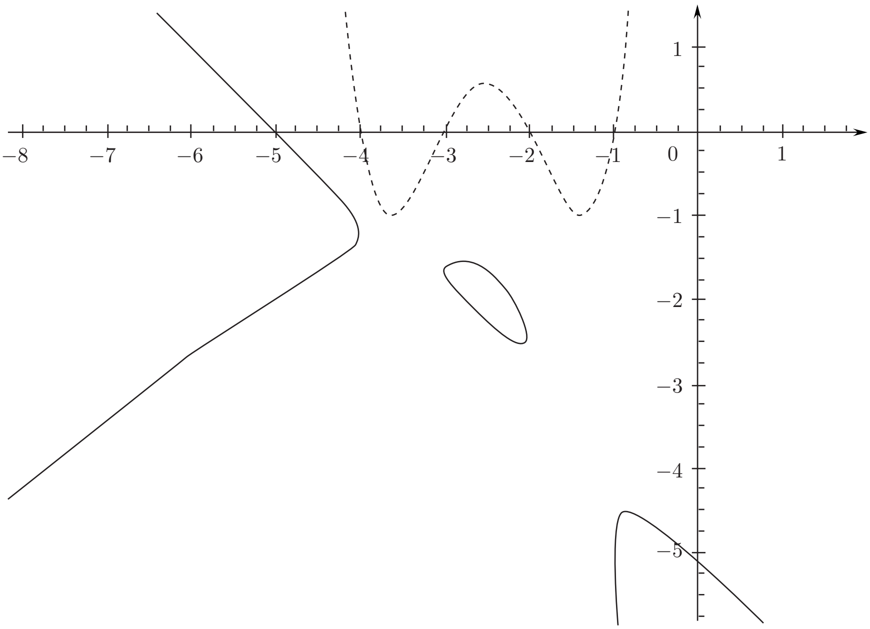


Figure 6.11
Detail of $xy^2 - \sqrt{4 \cdot 24}y = x^3 + 10x^2 + 35x + 50$ (solid line), and $p(x) = (x+1)(x+2)(x+3)(x+4) = x^4 + 10x^3 + 35x^2 + 50x + 24$ (dotted line). ©Niccolò Guicciardini.

Stirling obtained two values for the ordinate:

$$y = \frac{e}{x} + \frac{d}{e} + Ax + Bx^3 + \cdots \tag{6.27}$$

$$y = -\frac{d}{e} - Ax - Bx^2 - \cdots . \tag{6.28}$$

From equation (6.27) one immediately deduces that the axis of the ordinates is an asymptote. Of course, the axis of the ordinates is cut by the curve at $y = -d/e$. In correspondence with this asymptote, there are "two infinite legs of the curve, one on each side in contrary directions."[47]

If x is made to continue infinitely in a positive direction, there will always be a positive and a negative value for y "increasing without limit." Hence, Stirling inferred the existence of two more infinite legs.[48]

[47] "indicat ordinatam primam esse Asymptoton, & habere duo crura ad diversas ejus partes posita, & in plagas oppositas tendentia." Ibid., p. 89.
[48] "augebuntur simul ordinatae valores sine limite." Ibid., p. 88.

If x is negative, the equation becomes

$$-xy^2 - ey = -ax^3 + bx^2 - cx + d. \tag{6.29}$$

In this case, too, when x is made to continue infinitely, there will always be a positive and a negative infinitely increasing values for y. The number of infinite legs, therefore, will be six.

In order to study the behavior of the curve at infinity, Stirling expanded y into a series that converges for large values of x. He obtained[49]

$$y = x\sqrt{a} + \frac{b}{2\sqrt{a}} + \frac{4ac - b^2 + 4ae\sqrt{a}}{8ax\sqrt{a}} + \cdots \tag{6.30}$$

$$y = -x\sqrt{a} - \frac{b}{2\sqrt{a}} - \frac{4ac - b^2 - 4ae\sqrt{a}}{8ax\sqrt{a}} + \cdots . \tag{6.31}$$

The first two terms of these series indicate the existence of two rectilinear asymptotes

$$y = \pm\sqrt{a}\left(x + \frac{b}{2a}\right), \tag{6.32}$$

cutting the axis of the abscissae at $x = -b/(2a)$, having inclinations equal to $\sqrt{a}$ and $-\sqrt{a}$, and cutting the axis of the ordinates at $y = \pm b/(2\sqrt{a})$.

Therefore, Stirling prescribed (i) to set $AD = b/(2a)$, and $Ad = A\delta = b/(2\sqrt{a})$; (ii) to join Dd, $D\delta$; and (iii) to produce Ad, Dd, $D\delta$, tracing the three asymptotes, which form a triangle $Dd\delta$.

Stirling's Proposition 16 is fundamental to the study of redundant hyperbolas. This class of cubics is further subdivided into genera and species in function of the values of the coefficients and the number and disposition of the roots of the polynomial $p(x) = ax^4 + bx^3 + cx^2 + dx + e^2/4$. One can only conjecture that

[49] Guardeño observed in his detailed commentary on the *Enumeratio* (Newton, *Análisis de Cantidades* (2003), p. 163),

$$\begin{aligned}\frac{\sqrt{ax^4 + bx^3 + cx^2 + dx + e^2/4}}{x} &= \sqrt{a}x\sqrt{1 + \frac{b}{a}\frac{1}{x} + \frac{c}{a}\frac{1}{x^2} + O(x^{-3})} = \\ \sqrt{a}x\left(1 + \frac{b}{2a}\frac{1}{x} + \left(\frac{c}{2a} - \frac{b^2}{8a^2}\right)\frac{1}{x^2} + O(x^{-3})\right) &= \sqrt{a}\left(x + \frac{b}{2a}\right) + \sqrt{a}\left(\frac{4ac - b^2}{8a^2}\right)\frac{1}{x} + O(x^{-2}).\end{aligned}$$

Note that

$$\left(1 + \frac{b}{a}\frac{1}{x} + \frac{c}{a}\frac{1}{x^2}\right)^{1/2} = 1 + \frac{1}{2}\left(\frac{b}{a}\frac{1}{x} + \frac{c}{a}\frac{1}{x^2}\right) + \frac{(1/2)(-1/2)}{2}\left(\frac{b^2}{a^2}\frac{1}{x^2} + O(x^{-3})\right).$$

Newton had already followed the algebraic procedure that Stirling reconstructed in his commentary.[50]

6.5.4 Publication Strategy

From a cursory look at Newton's study of cubic curves one can surmise that Newton probably applied the analytical parallelogram (§6.5.2) and power series expansions (§6.5.3).

Evidence that Newton used infinite series (and fluxions) in the study of cubics is scarce in his printed work. In *De Analysi* (1669), Newton studied the cubic $y^3+axy+x^2y-a^3-2x^3=0$: after expanding y as a power series in x, he explained how this series can be used to determine the asymptotes of the cubic.[51]

In the surviving manuscripts related to the *Enumeratio*, proofs of most statements concerning cubics are missing. In order to reconstruct Newton's arguments one has to rely on the evidence available in very rare drafts of calculations that have survived, where the evidence is that he used series and fluxions. The messy appearance of one of these calculations dating from the late 1670s, which from a modern point of view would be quite important, when compared to the careful hand in which Newton wrote his demonstrations on porisms in the contemporary "Veterum Loca Solida Restituta," now considered a geometrical backwater, reveals the value that he attributed to algorithm at this moment in his career.[52] Infinite series were deployed on scraps of paper in order to reach results that were later geometrically laid down in carefully drafted manuscripts.

Even though the algebraic objects Newton was considering—third-degree polynomials—were very much part of the Cartesian canon, he manipulated them with tools—infinite series—that stretched far beyond Cartesian methods. Newton surpassed Cartesian methodology by freely manipulating the infinite (both in his projective geometry and his algebraic calculations). It is to these innovative algebraic tools that part III is devoted.

[50] Stirling's demonstration is in *Lineae Tertii Ordinis Neutonianae* (1717), pp. 87–9, and is reproduced by Talbot in his commentary to Newton, *Enumeration of Lines of the Third Order* (1861), pp. 49–51.
[51] MP, 2, pp. 226–8.
[52] See the rough draft calculations for the determination of the root x of a cubic equation by an infinite series in powers of y^{-1}. As Whiteside noted in his commentary, here Newton deployed the method of "resolution of affected equations" that he had elaborated in *De Methodis* (1671) (see §7.5). Add. 3961.1, f. 22v. MP, 4, p. 389

III New Analysis and the Synthetic Method

Part III shifts from common analysis to new analysis, namely, Newton's method of series and fluxions. The new analysis was understood by Newton's contemporaries as an algebraic method that implied the use of infinite series and infinitesimally small magnitudes, whereas common analysis coincided with methods confined to algebraic equations. The time span is from 1669 (date of composition of *De Analysi*) to the early 1690s (date of composition of *De Quadratura*). In his youth Newton enthusiastically endorsed the new analysis of the moderns. However, around 1670 he began to seek some firmer ground on which to establish his analytical method of series and fluxions. Newton thus developed a synthetic method of fluxions, which he saw as consonant with ancient mathematical practice.

Chapter 7 considers the method of series developed in *De Analysi*, in which Newton presented himself to Collins as a creative analyst capable of manipulating infinite series very much in accordance with the program delineated by Wallis. Chapter 8 focuses on the analytical method of fluxions, both direct and inverse, elaborated in *De Methodis*, and in a more advanced form in *De Quadratura*. Chapter 9 turns to Newton's synthetic version of the method of fluxions as provided in "Geometria Curvilinea," the *Principia*, and the introduction to *De Quadratura*. With this mature version of the method Newton distanced himself from new analysis.

The synthetic version of the method of fluxions is based on postulates or lemmas concerning the limits of ratios and sums of "vanishing magnitudes"; its purpose is to allow the determination of tangents to curves and the calculation of curvilinear areas by geometrical arguments based on limiting procedures. A tension is apparent between Newton's desiderata and his mathematical practice. In particular, Newton's methods of quadrature, systematized in what nowadays (in Leibnizian terminology) would be called integral tables, are carried on in terms that are essentially algorithmic and difficult to reduce to synthetic form.

7 The Method of Series

> From all this it is to be seen how much the limits of analysis are enlarged by such infinite equations: in fact by their help analysis reaches, I might almost say, to all problems.
>
> — Isaac Newton, 1676

> The paradox remains that such Wallisian interpolation procedures, however plausible, are in no way a proof, and that a central tenet of Newton's mathematical method lacked any sort of rigorous justification.
>
> —Derek T. Whiteside, 1961

7.1 Wallis's *Arithmetica Infinitorum* (1656)

One of the key elements of Newton's new analysis is the use of infinite series, or infinite equations. The binomial theorem that Newton formulated in the winter of 1664–1665 is only the first step in the new analytical method that allowed him to handle mechanical curves and calculate the areas of curvilinear figures. These techniques are based on the kind of "inductive" generalizations that Newton found in the work of Wallis, most notably in *Arithmetica Infinitorum* (1656).[1] I turn to some salient features of this work before dealing with the method of series that Newton codified in *De Analysi* (completed in 1669) and the first pages of the *De Methodis* (completed in 1671).[2]

Wallis was Savilian Professor of Geometry in Oxford. His appointment in 1649 had been determined more by politics than by any mathematical achievement on his part. Wallis had served as secretary of the Assembly of Divines and as cryptographer had been of great help to Parliamentarians during the Civil War. As Scriba remarked, "[F]ew people in 1649 could have foreseen that within a few years the thirty-two-years-old theologian would become one of the leading mathematicians of his time."[3]

Epigraph sources: (1) Newton to Oldenburg for Leibniz (June 13, 1676), *Correspondence*, 2, p. 39. "Ex his videre est quantum fines Analyseos per hujusmodi infinitas aequationes ampliantur: quippe quae earum beneficio, ad omnia, pene dixerim, problemata ... sese extendit." *Correspondence*, 2, p. 29. (2) Whiteside, "Newton's Discovery of the General Binomial Theorem" (1961), p. 180.

[1] Printed by July 1655, published in Wallis, *Operum Mathematicorum, Pars Altera* (1656), pp. 1–199.

[2] See Abbreviations and Conventions for bibliographical details on *De Analysi* and *De Methodis*.

[3] Scriba, "John Wallis" (1976), p. 147.

Armed with a solid background in theology, metaphysics, and logic, Wallis began to enter cutting-edge mathematical research when, in 1647–1648, he came across William Oughtred's *Clavis*, which proved to be an important source for the young Newton, too. The pronounced symbolism of Oughtred's work fascinated Wallis, who remained a defender of the heuristic power of algebraic notation all his life. Most notably, in *De Sectionibus Conicis* (1655), he defined the conic sections in algebraic terms, strongly supporting the approach to geometry that stemmed from Descartes' *Géométrie*.[4] Jan de Witt, one of the Cartesian mathematicians in van Schooten's circle, did very much the same thing in *Elementa Curvarum Linearum* (completed in 1649), a treatise that appeared in the second Latin edition of Descartes' *Geometria* (1659–1661).[5]

Arithmetica Infinitorum (1656) instead deals with a topic that had remained untouched by Descartes: "the quadrature of curvilinear figures." This expression was used by seventeenth-century mathematicians to denote methods for the determination of either the area of a surface bound by a curve or the volume of a solid bound by curvilinear surfaces. This was a very important subject in the mid-seventeenth century, not least because of its applications to natural philosophy.

Wallis drew from Evangelista Torricelli's *Opera Geometrica* (1644), a work that expounded and extended Bonaventura Cavalieri's "geometry of indivisibles."[6] But while the Italians had developed a geometry of indivisibles, Wallis wished to propose an "arithmetic of indivisibles." Following Kepler, Torricelli, Pascal, and other early-seventeenth-century pioneers of quadrature methods, it was customary to conceive of a surface as an aggregate of lines or of infinitesimal parallelograms. The language employed to deal with such infinities was varied and posed enormous conceptual problems.[7] Wallis would have said that any plane surface can be seen as being composed of an infinite number of parallelograms of equal altitude, the altitude itself being denoted by a bewildering $1/\infty$.[8] Similar parlance was often accompanied by claims that the new nonrigorous methods could be reframed in terms of the more conventional exhaustion techniques illustrated in the works of Archimedes.

Wallis pointed out that it was possible, for instance, to envisage the parabolic surface (figure 7.1) as being composed of infinitely many infinitesimal parallelo-

---

[4] Wallis, *De Sectionibus Conicis, Nova Methodo Expositis Tractatus* in *Operum Mathematicorum Pars Altera* (1656).

[5] de Witt, *Elementa Curvarum Linearum*, in R. Descartes, *Geometria* (1659–61), pp. 153–340.

[6] Cavalieri's method was fundamentally changed by Torricelli into a different theory; Cavalieri studiously avoided considering the continuum as composed of infinitesimals. On the relationships between Cavalieri and Torricelli, see Andersen, "Cavalieri's Method of Indivisibles" (1985); Giusti, *Bonaventura Cavalieri and the Theory of Indivisibles* (1980); De Gandt, *Force and Geometry in Newton's Principia* (1995), pp. 185–202.

[7] Blay, *Reasoning with the Infinite* (1998); Mancosu, *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century* (1996); Malet, *From Indivisibles to Infinitesimals* (1996).

[8] Stedall, "John Wallis, *Arithmetica Infinitorum*" (2005), p. 25.

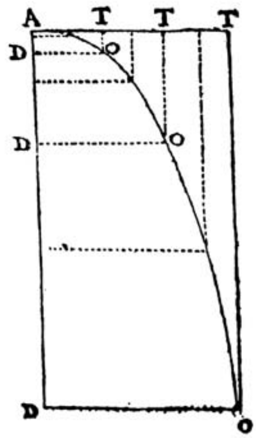


Figure 7.1
Parabolic surface in Wallis's *Arithmetica Infinitorum*. Source: Wallis, *Opera* (1695), 1, p. 383. Courtesy of the Biblioteca Angelo Mai (Bergamo).

grams. If a is the small distance between the ordinates, and $TO = (DO)^2$, then the ratio of the area of the curvilinear surface ATO to the area of the rectangle $ATOD$ is approximated by the arithmetical ratio

$$\frac{0^2 + a^2 + (2a)^2 + (3a)^2 + \ldots + (na)^2}{(na)^2 + (na)^2 + (na)^2 + (na)^2 + \ldots + (na)^2}, \tag{7.1}$$

calculated for large values of n (where n is a positive integer).[9] Wallis maintained that one could obtain the exact value of the ratio between the areas by letting a be equal to $1/\infty$, thus letting the number of terms in the numerator and denominator of ratio (7.1) be infinite (note that na is kept fixed).

Wallis calculated the exact value by what he termed an induction, whereby one considers the pattern that emerges from the ratio

$$\frac{0^2 + 1^2 + 2^2 + 3^2 + \ldots n^2}{n^2 + n^2 + n^2 + n^2 + \ldots + n^2}, \tag{7.2}$$

for increasing values of n. In this example one obtains

$$\frac{0+1}{1+1} = \frac{1}{3} + \frac{1}{6}$$

[9] Wallis would express the area of the parabolic surface ATO as being approximated by $0^2 + a \cdot a^2 + a \cdot (2a)^2 + a \cdot (3a)^2 + \ldots a \cdot (na)^2$, and the area of the rectangle $ATOD$ as $(na)^2 \cdot na$.

$$\begin{aligned}
\frac{0+1+4}{4+4+4} &= \frac{1}{3}+\frac{1}{12}\\
\frac{0+1+4+9}{9+9+9+9} &= \frac{1}{3}+\frac{1}{18}\\
\frac{0+1+4+9+16}{16+16+16+16+16} &= \frac{1}{3}+\frac{1}{24}\\
\ldots &= \ldots
\end{aligned}$$

For all n considered so far, the result is $1/3$ plus a fraction whose value becomes smaller as n increases. Wallis stated,

> The simplest way of investigating this and other problems is to set forth a certain number of cases and observe the resulting ratios, and then compare them with one another in order that the universal proposition can then be known by induction.[10]

Following his method of induction, Wallis concluded that the ratio (7.2) is equal to $1/3 + 1/6n$, for every positive integer n.[11]

Further, a limit argument allowed Wallis to state that the area of the parabolic surface ATO, conceived of as composed of an infinite number of rectangles of altitude $1/\infty$, is one-third of the area of the rectangular surface $ATOD$. This was a very well-known result, of course, but it was achieved according to an innovative method. It is interesting to consider how Wallis justified the limiting procedure:

> Since, moreover, as the number of terms increases, the excess over one third is continually decreased, in such a way that at length it becomes less than any assignable quantity (as is clear); if one continues to infinity, it will vanish completely.[12]

Newton was to justify similar limit procedures by mobilizing different conceptual resources, while employing a terminology analogous to that of Wallis (see chapter 9).

Wallis soon realized that this method could be applied to the squaring of higher-order parabolas. Thus he applied induction and limit arguments to ratios of the form

$$\frac{0^r + a^r + (2a)^r + (3a)^r + \ldots + (na)^r}{(na)^r + (na)^r + (na)^r + (na)^r + \ldots + (na)^r}, \tag{7.3}$$

where $r = 3, 4$, etc. Again, Wallis obtained well-known results. When $r = 3$, the area under the parabola of equation $y = x^3$ calculated between 0 and b is $b^4/4$; when

[10] Wallis, *Opera*, 1, p. 365. Translation in Jesseph, *Squaring the Circle* (1999), p. 176.

[11] Wallis's method should be distinguished from the modern principle of mathematical induction formalized in the Dedekind-Peano axioms.

[12] Wallis, *The Arithmetic of Infinitesimals* (2004), p. 27. "Cum autem crescente numero terminorum, excessus ille supra ratione subtriplam ita continuo minuatur, ut tandem quolibet assignabili minor evadat, (ut patet;) si in infinitum procedatur, prorsus evaniturus est." Wallis, *Arithmetica Infinitorum* (1656), p. 16.

$r = 4$, the area is $b^5/5$. This pattern was extended to all positive r by appealing to the principle of induction.

Wallis further extended this result to r fractional. He was the first to extend the rule $a^p a^q = a^{p+q}$ "by analogy" to negative and fractional exponents, even though he did not explicitly use the notation for fractional powers. Wallis began to extend the pattern emerging for positive integers to some fractional exponents by interpolating numerical tables. He convinced himself that the pattern was still valid and extended it by induction to all fractional powers or even real exponents. In modern terms, what Wallis achieved was a result that we would express as

$$\int_0^b x^r \, dx = \frac{b^{r+1}}{r+1}, \tag{7.4}$$

for r real and $\neq -1$.

The fact that Wallis's result breaks down for $r = -1$ means that he could not solve an important geometrical problem: the squaring of the hyperbola $xy = 1$. But it is on the squaring of the circle that Wallis spent all his energy: indeed, *Arithmetica Infinitorum* was written with the purpose of solving this problem. Wallis began by subdividing the quadrant of a circle of radius R into an infinite number of parallelograms with equal bases $a = R/\infty$ and heights equal to $\sqrt{R^2 - (ka)^2}$ (figure 7.2). Without the aid of the binomial series, the attempt to determine and sum the areas of the parallelograms proved to be a *tour de force*. Wallis's technique of interpolation (as he called it) has often been described. One of the most accurate analyses is provided by Stedall, whose lead I follow in this section.[13] For convenience, I here employ modern notation. What Wallis did, essentially, was to

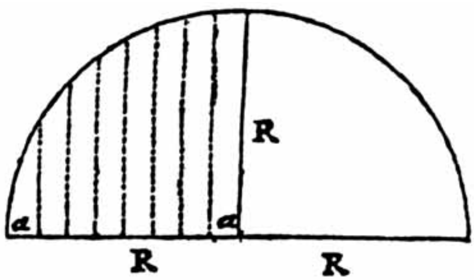


Figure 7.2

Quadrant of the circle subdivided into an infinite number of parallelograms in Wallis's *Arithmetica Infinitorum*. Source: Wallis, *Opera* (1695), 1, p. 417. Courtesy of the Biblioteca Angelo Mai (Bergamo).

[13] Stedall, *"A Discourse Concerning Algebra"* (2002), pp. 159–65.

tabulate the reciprocals of certain integrals:

$$f(p,q) = \frac{1}{\int_0^1 (1 - x^{1/p})^q dx}. \tag{7.5}$$

Wallis could calculate this for positive integer values of q and for fractional p. Of course, $f(1/2, 1/2) = 4/\pi$ is the value that Wallis tried to catch by interpolation. This proved to be terribly difficult, yet Wallis was able to identify two sequences of lower and upper bounds for $4/\pi$. In the end Wallis could state his celebrated continued product (returning here to his notation):

$$\frac{4}{\pi} = \frac{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times \&c.}{2 \times 4 \times 4 \times 6 \times 6 \times 8 \times \&c.} \tag{7.6}$$

Wallis handled infinitesimals, infinite series, and infinite products with bewildering prowess. One of Newton's most decisive starting points as a creative mathematician was *Arithmetica Infinitorum*. Before turning to Newton's use of series in representing curves, areas, and arclengths—a technique he understood as a means to cross the boundaries set by the Cartesian canon—it is worth turning to the reception of the Wallisian proof methods. Wallis had to withstand considerable criticism. Similar criticism would have been quite embarrassing for Newton, since he was defending mathematics as a means capable of injecting certainty into natural philosophy.

7.2 Criticism Leveled at Wallis

Wallis's mathematical work, most notably his *Arithmetica Infinitorum*, was the polemic target of Pierre de Fermat and Thomas Hobbes. We do not know how much Newton knew about the controversy between Hobbes and Wallis. He certainly knew about Fermat, since the letters of the French mathematician were reproduced in Wallis's *Commercium Epistolicum* (1658), which Newton read in his early days as a mathematician.

One of the criticisms leveled at Wallis concerned the validity of induction. The fact that a proposition is proven true for a few numbers belonging to a class does not imply that it is valid for all the members of the class, as Fermat, a master of number theory, knew too well.[14] In order to reject similar criticism, Wallis devised a well-structured reply. First, he claimed that induction methods were not his invention but had been employed both recently by Henry Briggs and Viète and in the ancient world by Euclid. So, Wallis claimed (and this part of his reasoning is less

[14] Fermat invited Wallis to devote himself to number theory, but Wallis found it of little interest. Number theory struck him as something of limited use in applications, in other words, as a useless inquiry.

convincing), even Euclid assumed that what has been proven for a single equilateral triangle can apply to all equilateral triangles. Second, Wallis insisted on the fact that in *Arithmetica Infinitorum* he wanted to provide a method of *discovery*, not *proof*. Wallis's strategy was to claim that his mathematical inductions were part of the analytical process of discovery. Further, he insisted that he had the highest regard for the rigorous methods of proof of the ancients and that his heuristic methods could be reframed in classical terms.

Hobbes, with whom Wallis had a notorious squabble,[15] and Fermat complained about Wallis's excessive reliance upon symbols. For instance, in 1657, Fermat stated,

> It is not that I do not approve it, but all his propositions could be proved in the usual, regular Archimedean way in many fewer words than his book contains. I do not know why he has preferred this method with algebraic notation to the older way which is both more convincing and more elegant.[16]

In 1658 Fermat addressed Wallis as follows:

> [W]e advise that you would lay aside (for some time at least) the Notes, Symbols, or Analytick Species (now since Vieta's time, in frequent use,) in the construction and demonstration of Geometrick Problems, and perform them in such method as Euclide and Apollonius were wont to do; that the neatness and elegance of Construction and Demonstration, by them so much affected, do not by any degrees grow into disuse.[17]

Note how Fermat praised the elegance, conciseness, and neatness of the geometrical constructions and demonstrations of Euclid, Apollonius, and Archimedes. By the 1670s, Newton had come to share Fermat's position (see chapters 4 and 5).

Fermat's criticisms have been often misinterpreted by historians puzzled by the fact that a leading figure of the analytical school (who employed algebra in the analysis of geometrical problems and made use of infinitesimal techniques) could attack Wallis because of his reliance on symbols. A similar bewilderment followed the publication of Newton's *Arithmetica Universalis*. How could Newton end a treatise devoted to algebra with an appendix that argued so strongly in favor of the autonomy and superiority of geometry? My answer, already suggested (§4.5 and §4.6), is

---

[15] The best account is Jesseph, *Squaring the Circle* (1999).

[16] Fermat to Kenelm Digby (August 15, 1657): "Ce n'est pas que je ne l'approuve, mais toutes ses propositions pouvant estre demonstrées via ordinaria legitima et Archimedea en beaucoup moins de parolles, que n'en contient son livre. Je ne sçay pas, pourquoy il à preferé cette maniere par notes Algebriques à l'ancienne, qui est & plus convainquante, & plus elegante." Wallis, *Commercium Epistolicum* (1658), letter 12. Translation in Stedall, *"A Discourse Concerning Algebra"* (2002), p. 170.

[17] Fermat to Digby (June 1658), in Wallis, *Commercium Epistolicum*, letter 46. Translated in Wallis, *A Treatise of Algebra* (1685), p. 305, and discussed in Stedall, *"A Discourse Concerning Algebra"* (2002), p. 171.

that one has to place these statements in the context of the canon of problem solving adopted by seventeenth-century mathematicians. It is in this context that Newton could maintain that symbolic algebra had no role to play in the constructive (synthetic) part of the canon, whereas it could be deployed in the resolutive (analytical) part. Similarly, Fermat urged Wallis to use geometry in constructions and demonstrations. That is why Wallis could frame a reply that both expressed admiration for the ancient synthetic methods and articulated a defense of the heuristic power of symbolism. Wallis emphasized that he valued the ancient method of construction, but that his work was about analysis, not synthesis. To criticize his work, as Fermat did, was to miss the point. Heavily relying on the widespread topos that attributed no heuristic power to the synthesis in which the ancients published their works, as well as on the myth that they had kept their methods of discovery concealed, Wallis wrote,

> To the Elegance and neatness of the Ancients way of Construction and Demonstration, I am not Enemy. And that these Propositions might be so demonstrated, I was far from being ignorant . . . .[Fermat] doth wholly mistake the design of that Treatise; which was not so much to shew a Method of Demonstrating things already known (which the Method that he commends, doth chiefly aim at,) as to shew a way of *Investigation* or finding out of things yet unknown: (Which the Ancients did studiously conceal). . . . And that therefore I rather deserve thanks, than blame, when I did not only prove to be true what I had found out; but shewed also, how I found it, and how others might (by those Methods) find the like.[18]

Wallis was so enthusiastic in his endorsement of arithmetic and algebra as the analytical tools the ancients were familiar with but had chosen to conceal, that he was criticized by many, most notably by Hobbes, with whom Wallis had a protracted quarrel adumbrated by political overtones. Wallis has often been portrayed as a defender of modernity against the reactionary nostalgia of geometrical purity. Yet, as Neal has shown, the cleavage between modernity and classicism cannot be drawn so simplistically, not only because many first-rank mathematicians who greatly contributed to the scientific revolution, such as Huygens, Barrow, and Newton, were critical of the use of symbols but also because Wallis defended symbolism on the basis of values that he located in the classical tradition.[19]

When Wallis was attacked because of his use of infinite products and infinitesimal magnitudes, he replied, as was customary in the middle of the seventeenth century, that his arithmetic of infinities was only an abbreviated, more direct version of the method of exhaustion. Wallis's position on the matter was spelled out

---

[18] Wallis, *A Treatise of Algebra* (1685), pp. 305–6. See Stedall, *"A Discourse Concerning Algebra"* (2002), p. 172.

[19] As Neal observed, Wallis was following the classical Aristotelian tradition when he distinguished pure (arithmetic and geometry) from mixed mathematics, and when he attributed his preference for arithmetic over geometry to the fact that the former is more abstract and universal than the latter. Neal, *From Discrete to Continuous* (2002), p. 153.

in detail in *Treatise of Algebra* (1685), a lengthy historical presentation of specious arithmetic (as treated by Oughtred) and of the quadrature methods of *Arithmetica Infinitorum*, written in the historicist style of Gerardus Joannes Voss's *De Universae Matheseos Natura et Constitutione* (1660) and Claude François Milliet Deschales' *Cursus seu Mundum Mathematicus* (1674).[20] In the historical reconstruction provided in *A Treatise of Algebra*, Wallis carefully presented results first according to Archimedean exhaustion, then according to what he identified as "Cavalieri's geometry of indivisibles,"[21] and finally according to his own arithmetic of indivisibles. Wallis stressed the continuity, rather than discontinuity, between the Greek tradition and *Arithmetica Infinitorum*. While in *Arithmetica Infinitorum* and in *De Sectionibus Conicis*, Wallis used infinities quite freely, in later works, such as *A Treatise of Algebra* and *A Defense of the Treatise of the Angle of Contact* (1684),[22] his defense of the use of infinitesimal magnitudes emphasized the continuity with exhaustion techniques.

The method of exhaustion, Wallis claimed in his *Treatise of Algebra*, is certainly rigorous. However, it is too cumbersome to be "parcelled out into several lemmas and preparatory Propositions. Which though it might look more August [is] less edifying."[23] Rather, Wallis argued, the arithmetic of infinities is more direct and simple. Yet, the principles on which it rests, Wallis continued, are the same as those assumed in Archimedean proofs. According to Wallis, one has to understand infinitesimals as variable magnitudes that can be made less than any assignable finite magnitude:

> [A]ll continual approaches, in which the Distance comes to be less than any assignable, must be supposed, if infinitely continued, to determine in a Coincidence or Concurrence. The Difference thus coming to nothing or (what Geometry accounts as such,) less than any assignable.[24]

Clearly, Wallis was trying to justify his limit arguments, which he had employed in the quadrature of curvilinear figures, in terms of a process of continual approach. Newton's method of first and ultimate ratios was expressed in a language reminiscent of Wallis's (see chapter 9).

Similarities with Newton also emerge in *A Defense of the Treatise of the Angle of Contact* (1684), particularly where Wallis talked about inchoative or inceptive

---

[20] On Wallis as a historian of mathematics, see Stedall, "Of Our Own Nation" (2001), and Scott, "John Wallis as Historian of Mathematics" (1936).

[21] Note, however, that this was Torricelli's version of Cavalieri's indivisibles.

[22] This tract was also appended to *A Treatise of Algebra* (1685), pp. 69–105. It is a defense against Vincent Leotaud of *De Angulo Contactus et Semicirculi Tractatus*, a work printed in 1656 together with *Arithmetica Infinitorum* in *Operum Mathematicorum Pars Altera*.

[23] Wallis, *A Treatise of Algebra* (1685), p. 305.

[24] Ibid., p. 284.

quantities. By this, Wallis meant that his indivisibles must be understood as generative of finite magnitudes. So a point is inceptive of a line conceived as generated by the motion of a point, a line is inceptive of a surface generated by the motion of a line, and acceleration is inceptive of velocity:

> There are some things, which tho' as to some kind of Magnitude, they are nothing; yet are in the next possibility of being somewhat. They are *not* it, but *tantum non*; they are in the next possibility to it; and the Beginning of it: Tho' not as *primum quod sit* (as the Schools speak) yet as *ultimum quod non*. And may very well be called *Inchoatives* or *Inceptives*, of that somewhat to which they are in such possibility.[25]

In a different context, Newton similarly attempted to defend the legitimacy of the use of moments (§9.4). Certainly, Newton proved extremely susceptible to the criticisms addressed against Wallisian techniques, since these techniques formed the backbone of his early mathematical discoveries. It is the use of the Wallisian techniques of induction and interpolation that in the mid-1660s led a young Newton to the discovery of the binomial series, a tool that allowed him to deal with mechanical curves and solve quadrature problems that lay beyond the boundaries of the Cartesian canon. However, his commitment to a program, spelled out in the early 1670s (see chapter 2), which invested mathematics with the role of injecting certainty into natural philosophy rendered Newton much more cautious than Wallis in the use and publication of the new analysis.[26]

7.3 The Binomial Series

> Newton in one stroke liberated mathematics from the concept of ratio and proportion that had pervaded all Greek and early European mathematical thinking and opened the way to perceiving areas (and associated logarithmic and trigonometrical quantities) as functions of a free variable.[27]

The young Newton enthusiastically endorsed the new analysis. It is thanks to a Wallisian interpolation technique that he began his creative mathematical career. In the winter of 1665, Newton came up with his first mathematical discovery: the binomial series for fractional powers. He began by attempting to calculate the areas of curvilinear surfaces subtended by the series of curves whose common base or axis

---

[25] Wallis, *A Defense of the Treatise of the Angle of Contact* (1684), cited in *A Treatise of Algebra* (1685), p. 96. See the commentary in Sellés, "Infinitesimals in the Foundations of Newton's Mechanics" (2006).

[26] For more information on Wallis's ideas concerning the philosophy of mathematics, see Malet, *From Indivisibles to Infinitesimals* (1996). A somewhat dated account of Wallis's mathematical work is in Scott, *The Mathematical Work of John Wallis* (1981). I recommend the cited works by Stedall. The Italian reader is now served by Maierù, *John Wallis* (2007).

[27] Stedall, *"A Discourse Concerning Algebra"* (2002), p. 177.

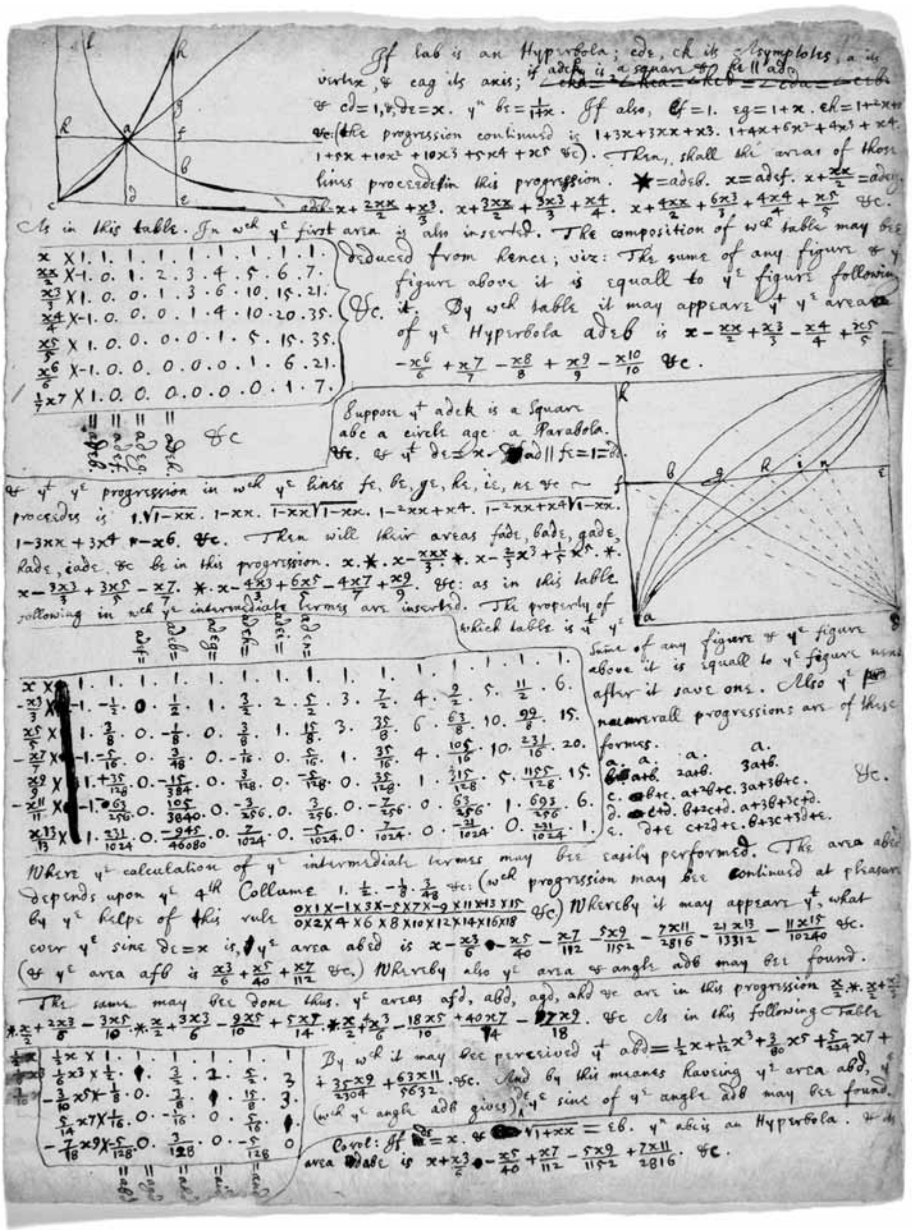


Figure 7.3
Attempted interpolations of Oughtred's Analyticall Table. Autumn 1665? Source: Add.3958.3, f. 72r. Reproduced by kind permission of the Syndics of Cambridge University Library.

is x and ordinates y are equal to $(1-x^2)^{0/2}$, $(1-x^2)^{1/2}$, $(1-x^2)^{2/2}$, $(1-x^2)^{3/2}$, $(1-x^2)^{4/2}$, $(1-x^2)^{5/2}$, and so on. Newton did not seek to calculate areas between fixed limits but left the upper bound as a free variable x. The area associated with the second curve in the preceding list would, of course, give the area of a circular segment, a much harder problem than the one tackled by Wallis, who sought to calculate the area of the circular quadrant. But it is exactly this generalization that allowed Newton to perceive an interesting pattern. By applying well-known results on quadrature of higher-order parabolas, Newton noted that the first, third, fifth, seventh, etc., curves subtend surfaces whose areas, calculated from 0 to x, are

$$\begin{aligned}
&x,\\
&x-\frac{1}{3}x^3,\\
&x-\frac{2}{3}x^3+\frac{1}{5}x^5,\\
&x-\frac{3}{3}x^3+\frac{3}{5}x^5-\frac{1}{7}x^7,\\
&\quad\dots
\end{aligned}$$

Newton noted that it is possible to rewrite these coefficients in terms of what he called Oughtred's Analyticall Table (figure 7.3), that is, the Pascal triangle. Newton tabulated this finding as follows:

n=	0	2	4	6	8	...	times
	1	1	1	1	1	...	x
	0	1	2	3	4	...	$-x^3/3$
	0	0	1	3	6	...	$x^5/5$
	0	0	0	1	4	...	$-x^7/7$
	0	0	0	0	1	...	$x^9/9$
						$\vdots$	$\vdots$

This table gives the area of the surface subtended by $y=(1-x^2)^{n/2}$ for $n=0,2,4,6,\dots$. The law of formation of the coefficients can immediately yield the coefficients for n even negative. This extrapolation, based on a typically Wallisian technique, whereby a pattern valid for a certain domain of numbers is extended by analogy to other domains, produces the following table:

n=	...	-4	-2	0	2	4	6	8	...	times
	...	1	1	1	1	1	1	1	...	x
	...	-2	-1	0	1	2	3	4	...	$-x^3/3$
	...	3	1	0	0	1	3	6	...	$x^5/5$
	...	-4	-1	0	0	0	1	4	...	$-x^7/7$
	...	5	1	0	0	0	0	1	...	$x^9/9$
	$\vdots$								$\vdots$	$\vdots$

This was a remarkable result, which was of great importance for Newton and his contemporaries. Newton then attempted to extend his table to n odd.

The process of discovery of the coefficient for n odd was rather intricate. After much effort, Newton was able to discern a pattern valid for n even, and he extended it by analogy, to n odd. He noted that the "intermediate termes ... are of this nature":[28]

a	a	a	a	a	...
b	b+c	b+2c	b+3c	b+4c	...
d	d+e	d+2e+f	d +3e+3f	d+4e+6f	...
g	g+h	g+2h+i	g+3h+3i+k	g+4h+6i+4k	...
...	...	...	...	...	...

When Newton explained this to Leibniz in 1676, he provided a simpler account. The two procedures are nevertheless equivalent.[29] One can express Newton's account in modern notation by saying that when n is even, say, $n = 2m$, the nth column is such that its kth entry ($k = 0, 1, 2, 3, \ldots$) is

$$m!/k!(m-k)!. \tag{7.7}$$

By means of Wallisian analogy Newton extended this relation to the odd columns. The result is the following table:[30]

n=	-4	-3	-2	-1	0	1	2	3	4	...	times
...	1	1	1	1	1	1	1	1	1	...	x
...	-2	$-3/2$	-1	$-1/2$	0	$1/2$	1	$3/2$	2	...	$-x^3/3$
...	3	$15/8$	1	$3/8$	0	$-1/8$	0	$3/8$	1	...	$x^5/5$
...	-4	$-35/16$	-1	$-5/16$	0	$1/16$	0	$-1/16$	0	...	$-x^7/7$
...	5	$315/128$	1	$35/128$	0	$-5/128$	0	$3/128$	0	...	$x^9/9$
$\vdots$										$\vdots$	$\vdots$

Newton could now write the area subtended to the curve $y = (1-x^2)^{1/2}$ as

$$x + 1/2(-x^3/3) - 1/8(x^5/5) + 1/16(-x^7/7) - 5/128(x^9/9) + \ldots . \tag{7.8}$$

[28] MP, 1, p. 130.

[29] "I found that on putting m for the second figure, the rest would be produced by continual multiplication of the terms of this series: $\frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$, etc." Newton, *Correspondence*, 2, p. 130. English translation by Turnbull.

[30] MP, 1, p. 132. A clear analysis is provided in Whiteside, "Newton's Discovery of the General Binomial Theorem" (1961).

This is the result Newton needed to square the circle segment (figure 7.4). By subtracting the area of the triangle OPQ from the area of the segment $ORQP$, he could calculate the area of the circular sector OQR and therefore obtain an estimate of π (e.g., when $x = 1/2$, $\theta = \pi/6$).

Newton further noted that, since the area under $y = x^n$ and over the interval $[0, x]$ is $x^{n+1}/(n+1)$, one could extend the result valid for the area to the curve itself and state

$$(1-x^2)^{1/2} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 - \cdots . \tag{7.9}$$

Here he relied upon the intuition that the area of a curve represented by an infinite series is equal to the sum of the areas of each term. He was to formalize this as Rule 2 of *De Analysi* (§7.4).

A research on the circle quadrature based on Wallisian techniques thus led Newton to the binomial series that using modern notation we would express as

$$(a+x)^{m/n} = a^{m/n} + \frac{m}{n}a^{m/n-1}x + \frac{1}{1.2}\frac{m}{n}(\frac{m}{n} - 1)a^{m/n-2}x^2 + \cdots . \tag{7.10}$$

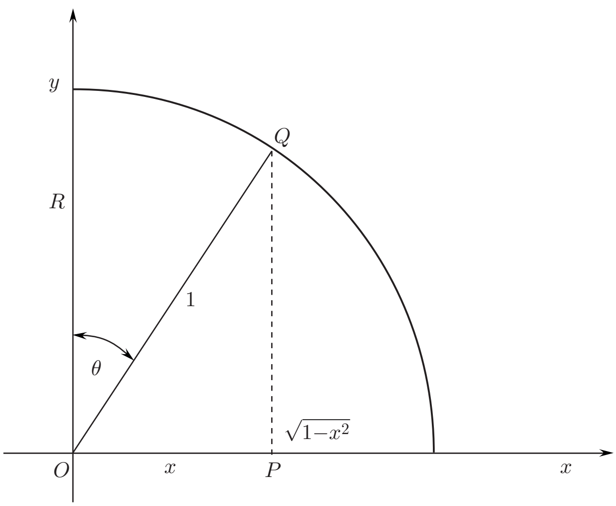


Figure 7.4
Newton was interested in calculating the area of the circle's segment $ORQP$ from 0 to x. The circle has radius 1 and equation $y = \pm\sqrt{1-x^2}$. Source: Edwards, *The Historical Development of the Calculus* (1979), p. 206. ©1979 Springer-Verlag New York Inc. With kind permission of Springer Science and Business Media.

Application of the binomial series to negative exponents leads to interesting results that had escaped Wallis.[31] Most notably, Newton wrote,

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots, \tag{7.11}$$

a result that he considered valid when x is small. Newton studied the hyperbola $y = (1+x)^{-1}$ for $x > -1$. He knew that the area under the hyperbola and over the interval $[0, x]$ for $x > 0$ (and the negative of this area when $-1 < x < 0$) is $\ln(1+x)$.[32] Applying Rule 2 (term-wise integration, in Leibnizian terminology) to series (7.11), Newton could express $\ln(1+x)$ as a power series:

$$x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - \cdots. \tag{7.12}$$

Newton somewhat intuitively understood that this series converges for $|x| < 1$ and $x = 1$. In this period questions regarding the convergence of infinite series were approached without any general theory of convergence. Mathematicians were simply happy to verify by application to numerical examples that the series (7.12) converged when the absolute value of x was smaller than 1. These series allowed Newton to calculate logarithms: he extended his numerical calculations to over fifty decimal places!

Since the proof of the binomial series rested on shaky inductive Wallisian procedures, Newton felt the need to verify the agreement of the series obtained by applying the binomial series with algebraic and numerical procedures. For instance, to $(1-x^2)^{1/2}$ he applied standard techniques of root extraction, and to $(1+x)^{-1}$ standard techniques of long division, and was happy to see that he obtained the first terms of series (7.9) and (7.11).

Three aspects of Newton's work on the binomial series are worth noting. First, Newton, following Wallis's suggestion, introduced negative and fractional exponents. Without this innovative notation ($x^{a/b}$ for $\sqrt[b]{x^a}$) no interpolation or extrapolation of the binomial series from positive integers to rationals would have been

[31] MP, 1, pp. 112–5; 134–42.

[32] It is difficult to determine how and when Newton came to realize this fact, which was first published by Grégoire de Saint Vincent in 1647 and more explicitly by Alphonse Antonio de Sarasa in 1649. Even though explicit reference to logarithms is absent in the early calculations (c. 1665), it is likely that Newton carried them out in order to calculate logarithms. Wallis, in the *dedicatio* of his *Arithmetica Infinitorum* (1656), which Newton knew, mentions Grégoire's *Opus Geometricum* (1647), a book that is listed in Isaac Barrow's library, to which Newton had regular access. See Feingold, "Isaac Barrow's Library" (1990). Perhaps Newton read Grégoire's or Sarasa's works, or independently discovered the relation between hyperbolic areas and logarithms. This fact was deployed by Mercator in *Logarithmotechnia* (1668), p. 34, which Newton most probably read in late 1668 or early 1669 (see *Correspondence*, 2, p. 114). See also Add. 4004, ff. 80r–81v in MP, 2, pp. 184–9, the dating of which, however, is uncertain (1667? according to Whiteside). For a discussion, see Panza, *Newton et les Origines de l'Analyse: 1664–1666* (2005), pp. 166–70.

possible, as Serfati argued in *La Révolution Symbolique* (2005). Second, Newton obtained a method for representing a large class of curves by means of power series. According to Newton, curves are thus expressed not only by finite algebraic equations (as Descartes maintained) but also by infinite series (even fractional power series), understood by Newton and by his contemporaries as *infinite equations*. In 1665 mathematicians had just begun to appreciate the usefulness of infinite series as representations of difficult curves. Contrary to what Descartes had stated, mechanical curves, such as the logarithmic curve, can therefore receive an algebraic representation to which the rules of algebra can be applied. Before the advent of infinite series such curves had no algebraic representation but were generally defined in geometrical terms. Finally, it should be noted that Newton had a rather intuitive concept of convergence. For instance, while he realized that the series (7.12) can be applied when x is small, he developed no rigorous treatment of convergence.[33]

7.4 Infinite Series and Quadratures

The binomial series allowed Newton to access what according to Cartesian standards was forbidden territory.[34] Through a simple application of the binomial series Newton could express trigonometric relations and calculate the area subtended to a mechanical curve, the cycloid. These calculations are found in *De Analysi* (1669), the small treatise in which Newton systematized his results on the resolution of quadrature problems via infinite series. Barrow was so impressed by this tract that he immediately sent it to John Collins (§1.1).

Newton began *De Analysi* by enunciating three rules:

Rule 1 If C is a curve with Cartesian equation $y = ax^{m/n}$, then the area subtended under C and calculated from 0 to x is $an/(m+n)x^{(m+n)/n}$.

Rule 2 If y is equal to the sum of more terms (also an infinite number of terms), $y = y_1 + y_2 + \cdots$, then the area of the surface subtended under y is equal to the sum of the areas calculated for the terms y_1, y_2, $\ldots$.

Rule 3 In order to calculate the area of the surface subtended under a curve whose equation is $f(x, y) = 0$, one should expand y as a sum of terms of the form $ax^{m/n}$ and apply Rule 1 and Rule 2.[35]

[33] Note, however, that at the very end of *De Analysi*, Newton referred to *Elements*, X, 1, that is, to the basic proposition for the method of exhaustion, in order to provide a proof that "when x is small enough, the more the quotient is extracted the more it approaches to the truth, to the end that its difference from the exact value of y shall come to be less than any given quantity you please and that the quotient extended to infinity shall be equal to y." MP, 2, p. 245.

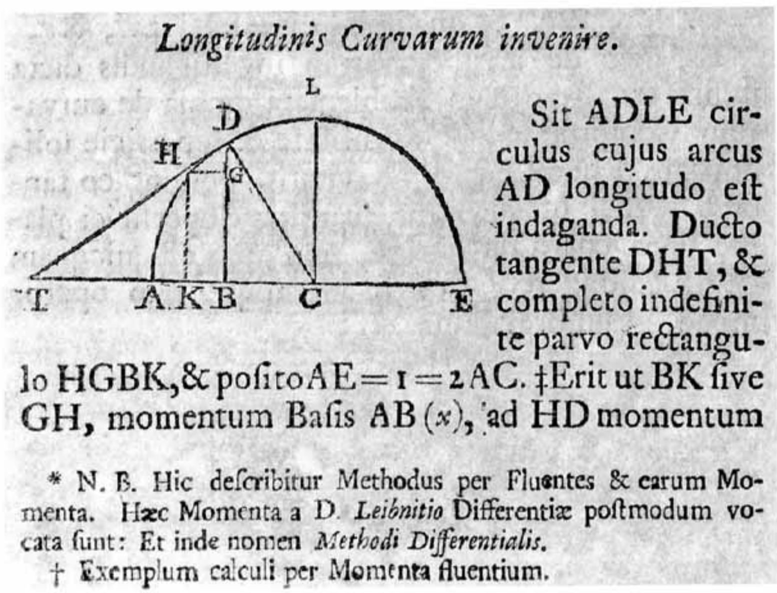
[34] In this section I am indebted to Edwards's clear presentation in *The Historical Development of the Calculus* (1979), pp. 189–230.

[35] See MP, 2, pp. 206–32. Most of these pages are occupied by Rule 3, the so-called resolution of affected equations (§7.5).

Rule 1 is a generalization—as previously noted, the upper bound is variable for Newton (§7.3)—of results stated by Wallis. Newton provided a proof of this rule based on fundamental concepts of the method of fluxions (§8.2.6). The binomial theorem proved to be an important tool for the implementation of Rule 3. In several cases, however, the binomial series cannot be applied. Newton therefore devised several clever techniques for "resolving affected equations" (§7.5), that is, expanding y as a fractional power series in x.

In *De Analysi*, Newton applied these rules to several quadrature problems, for instance, the following: Given a circle $ADLE$, determine the arclength AD (figure 7.5). First, Newton set $AB = x$, and considered the semicircle with diameter $AE = 1$ defined by equation $y = \sqrt{x - x^2}$. He considered the "moment of the base" GH and the "moment of the arc" HD.

As discussed in chapter 8, moments are the infinitesimal increments of a variable quantity. GHD is thus a triangle with infinitely small sides. From the similarity of triangles GHD, BTD, and BDC it follows that the ratio of the moment of the arc


Longitudinis Curvarum invenire.

Sit ADLE circulus cujus arcus AD longitudo eſt indaganda. Ducto tangente DHT, & completo indefinite parvo rectangulo HGBK, & poſito AE = 1 = 2 AC. ‡Erit ut BK ſive GH, momentum Baſis AB (x), ad HD momentum

* N. B. Hic deſcribitur Methodus per Fluentes & earum Momenta. Hæc Momenta a D. *Leibnitio* Differentiæ poſtmodum vocata ſunt: Et inde nomen *Methodi Differentialis.*

† Exemplum calculi per Momenta fluentium.

Figure 7.5

A quadrature from *De Analysi* (1669) reproduced in *Commercium Epistolicum*. One of its footnotes states, "Here the method of fluents and their moments is described. Subsequently these moments have been called differences by Leibniz: and therefore the name *differential method*," and that [this arclength calculation is] "[a]n example of calculation by the moments of fluents." Source: *Commercium Epistolicum* (1722), p. 84. Courtesy of the Biblioteca Nazionale Braidense (Milan).

AD to the moment of the base AB (that is, HD/GH) is equal to $\sqrt{x-x^2}/(2x-2x^2)$.[36] By expanding $\sqrt{x-x^2}/(2x-2x^2)$ via the binomial theorem into an infinite power series and applying Rule 3 (integrating term-wise), Newton obtained the arclength AD of the circle as $x^{1/2}(1+x/6+3x^2/40+5x^3/112+\dots)$. Next, he observed that, by choosing the coordinates so that $CB = x$ and setting the radius $CA = 1$, the arc $LD = z$ of the semicircle of equation $y = \sqrt{1-x^2}$ is given by

$$z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 \dots . \tag{7.13}$$

This is the series for $z = \arcsin x$. Note that the interval of convergence was verified by hand rather than by theory. Newton could obtain the series for $x = \sin z$ by what he called the method of the reversion of series.[37]

Newton applied the series for arcsin to the quadrature of the cycloid as follows. In *De Analysi*, he considered the cycloid generated by a circle with diameter $AH = 1$ (figures 7.6 and 7.7). The cycloid satisfies the well-known geometrical relation

$$BD = BK + \widehat{AK}, \tag{7.14}$$

so that the length of the rectilinear segment DK is equal to the length of the circular arc $\widehat{AK}$.

If one sets $AB = x$, the ordinate $BK = \sqrt{x-x^2}$ can be calculated by expanding the square root via the binomial theorem:

$$BK = x^{1/2} - \frac{1}{2}x^{3/2} - \frac{1}{8}x^{5/2} - \frac{1}{16}x^{7/2} \dots . \tag{7.15}$$

The circular arc $\widehat{AK}$ can be calculated by means of the series for arcsin, since $\widehat{AK} = \theta/2 = \arcsin AK = \arcsin \sqrt{x}$. Thus,

$$\widehat{AK} = x^{1/2} + \frac{1}{6}x^{3/2} + \frac{3}{40}x^{5/2} + \frac{5}{112}x^{7/2} \dots . \tag{7.16}$$

Thus, the ordinate BD of the cycloid, a mechanical curve excluded by Descartes, can symbolically be expressed as

$$BD = BK + \widehat{AK} = 2x^{1/2} - \frac{1}{3}x^{3/2} - \frac{1}{20}x^{5/2} - \frac{1}{56}x^{7/2} \ \dots . \tag{7.17}$$

[36] Indeed, $HD : GH = TD : BT = DC : BD = \frac{1}{2} : \sqrt{x-x^2} = \sqrt{x-x^2} : (2x-2x^2)$.

[37] The reversion of series was an application of the method of resolution of affected equations (§7.5).

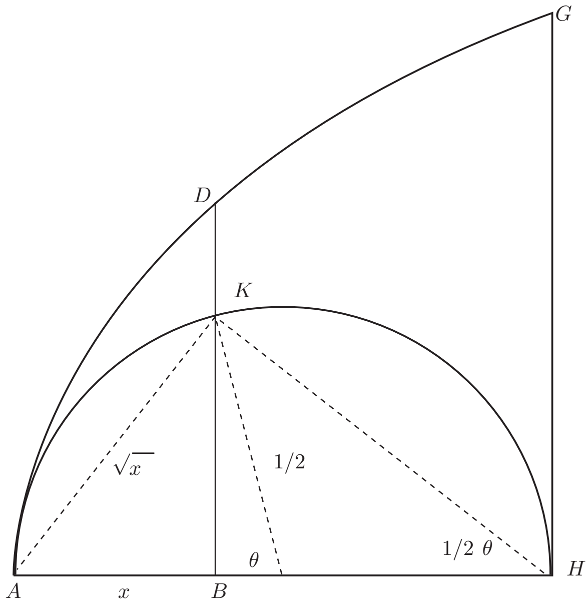


Figure 7.6
Cycloid. Note that the chord $AK = \sqrt{x}$, since $AK^2 = AB^2 + BK^2 = x$. Further, $AK = \sin(\theta/2)$. Source: Edwards, *The Historical Development of the Calculus* (1979), p. 207. ©1979 Springer-Verlag New York Inc. With kind permission of Springer Science and Business Media.

By applying term-wise Wallis's results for the area under the curve $y = x^{m/n}$ (in Leibnizian terms, $\int x^{m/n}dx = n/(m+n)x^{(m+n)/n} + C$), one finds that the area ABD subtended by the cycloid is

$$ABD = \frac{4}{3}x^{3/2} - \frac{2}{15}x^{5/2} - \frac{1}{70}x^{7/2} - \frac{1}{252}x^{9/2} \ \ldots . \tag{7.18}$$

All this is rather magical and must have impressed the young Newton, still an advocate of the new analysis of the moderns. Newton realized that mechanical curves and the calculation of their areas and arclengths could be dealt with by means of a symbolic calculation that only implied expansion into power series and term-wise integration (in Leibnizian terms). But infinite series proved to be an indispensable tool for studying algebraic curves, too (see chapter 6). The next section is devoted

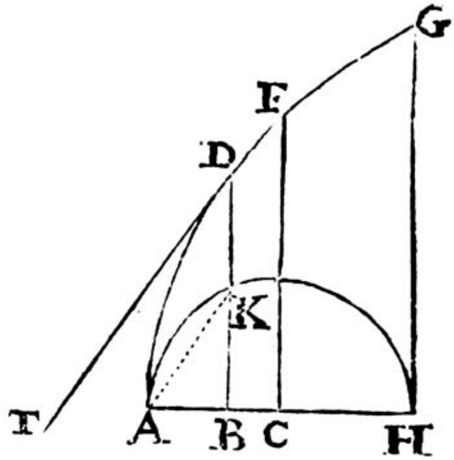


Figure 7.7
Cycloid in *De Analysi* (1669). Source: Newton, *Analysis per Quantitatum* (1711), p. 18. Courtesy of the Biblioteca Universitaria di Bologna.

to what Newton called the resolution of affected equations, namely, polynomial equations in two variables.

7.5 Resolution of Affected Equations

I make a preliminary observation here using modern notation. In order to implement Rule 3, Newton, given an algebraic equation $f(x, y) = \sum_{i,j} a_{i,j} x^i y^j = 0$, sought to express y as a series such that $y = \sum_{k=0}^{\infty} b_k x^{\alpha_k}$, where α_k is a rational number. Newton called this procedure the resolution of affected equations.

The method for the resolution of affected equations can be found at the beginning of *De Analysi* (1669) and *De Methodis* (1671).[38] It was also printed by Wallis in Chapter 94 of his *Algebra* after a transcript made by John Collins. This algebraic method is related to an iterative algorithm for approximating the real roots of algebraic equations in one unknown, a well-known procedure that was further developed by Joseph Raphson and by Thomas Simpson and amounts to what nowadays is called the Newton-Raphson method of approximation (see figure 7.10).[39]

The method for the resolution of affected equations was later systematized and generalized by Victor Puiseux and is often described in textbooks on algebraic

[38] See MP, 2, pp. 218–32 and MP, 3, pp. 48–71.
[39] Kollerstrom, "Thomas Simpson and 'Newton's Method of Approximation'" (1992).

geometry.[40] Compared to modern presentations, Newton's is far more based on concrete examples. His algebraic methods are a craft more than a theory, as is often evident in what follows.

The following example from *De Methodis* was also proposed in a famous letter that Newton addressed to Leibniz in 1676.[41]

The problem is that of studying a curve defined by the equation

$$y^6 - 5xy^5 + (x^3/a)y^4 - 7a^2x^2y^2 + 6a^3x^3 + b^2x^4 = 0, \tag{7.19}$$

in the vicinity of the origin, where the curve has a branching behavior. Hence, the need to expand y into a fractional power series in x.[42]

Which terms of equation (7.19) can be neglected in the vicinity of the origin? Recall that in section §6.5 I surmised that Newton knew how to answer this question when tracing the graphs of cubic curves.

In *De Methodis* and in his letter to Leibniz, Newton drew a grid where he located each monomial $x^m y^n$ (figure 7.8). If two monomials are connected on the grid, sometimes called Newton's analytical parallelogram, with a straight line, all the monomials intercepted by that straight line will be in geometrical progression.

B Fig. I.

x^4	x^4y	x^4y^2	x^4y^3	x^4y^4
x^3	x^3y	x^3y^2	x^3y^3	x^3y^4
x^2	x^2y	x^2y^2	x^2y^3	x^2y^4
x	xy	xy^2	xy^3	xy^4
0	y	y^2	y^3	y^4

A C

Figure 7.8
Newton's analytical parallelogram from the *epistola posterior*, which was sent on October 24, 1676, to Oldenburg for Leibniz. Source: Newton, *Analysis per Quantitatum* (1711), p. 31. Courtesy of the Biblioteca Universitaria di Bologna.

[40] See, e.g., Brieskorn and Knörrer, *Plane Algebraic Curves* (1986), pp. 370ff. Puiseux, "Recherches sur les Fonctions Algébriques" (1850). A detailed historical presentation can be found in Barbin, *A History of Algorithms* (1999), pp. 169–77.

[41] MP, 3, pp. 52–3. This example occurs in the *epistola posterior* to Leibniz (see chapter 16). See Coolidge, *A History of Geometrical Methods* (2003), p. 198, for further details. *Correspondence*, 2, pp. 126–7.

[42] It is interesting to note that Newton introduced the constants a and b with the purpose of maintaining dimensional homogeneity between the terms of the equation.

This fact leads to a simple rule for obtaining simpler equations that approximate a polynomial equation either close to the origin or at infinity.[43]

Mark the squares in the grid occupied by the monomials occurring in equation (7.19) to obtain a scheme like the one shown in figure 7.9. Newton called it a polygon, as he imagined to connect the asterisks with segments so as to obtain a convex polygon outside of which no asterisk lies.

Fig. 2.

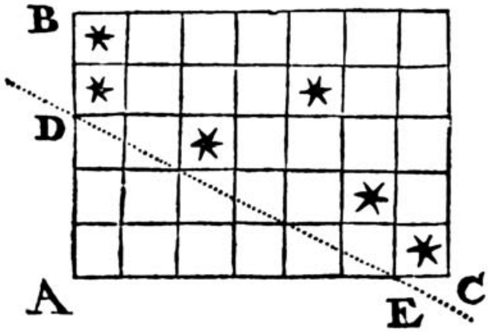


Figure 7.9
The polygon for equation (7.19): $y^6 - 5xy^5 + (x^3/a)y^4 - 7a^2x^2y^2 + 6a^3x^3 + b^2x^4 = 0$. One marks the squares corresponding to the terms of the equation with asterisks. Next, one joins the asterisks so as to obtain a convex polygon outside of which no asterisk lies. The dotted line is the prolongation of one of the sides of the polygon and identifies the terms of equation (7.20), $y^6 - 7a^2x^2y^2 + 6a^3x^3 = 0$, which approximates the curve near the origin. The other sides of the polygon are also interesting. The side connecting the asterisks associated with the monomials x^4 and x^3y^4 identifies the terms $(x^3/a)y^4$ and b^2x^4. Setting $y^4/x = k$, for some constant k, all the terms associated with the other asterisks are of lower degree when one expresses all the terms of the analytical parallelogram as powers of either x or of y. For instance, $7a^2x^2y^2 = 7a^2x^2\sqrt{kx} = 7a^2(y^8/k^2)y^2$ is of lower degree than $(x^3/a)y^4 = (x^3/a)kx = (y^{12}/(ak^3))y^4$. The side connecting the asterisks associated with the monomials x^3y^4 and y^6 identifies the terms $(x^3/a)y^4$ and y^6. Setting $x^3/y^2 = k$, all the terms associated with the other asterisks are again of lower degree. The two equations $(x^3/a)y^4 + b^2x^4 = 0$ and $x^3y^4 + y^6 = 0$ approximate the curve asymptotically, for $x \to -\infty$ and $y \to \pm\infty$. Finally, the side connecting the asterisks associated with the positions x^3 and x^4 of the analytical parallelogram identifies the terms of equation (7.19) when one sets $y = 0$. The equation corresponding to this last simple case is $6a^3x^3 + b^2x^4 = 0$; its roots are the points of intersection with the x-axis (in this case, $x = 0$ and $x = -6a^3/b^2$). Source: Newton, *Analysis per Quantitatum* (1711), p. 32. Courtesy of the Biblioteca Universitaria di Bologna.

[43] This method is described by Talbot in Newton, *Enumeration of Lines of the Third Order* (1861), pp. 88ff, and in Frost, *An Elementary Treatise on Curve Tracing* (2004), pp. 117–32.

Consider the dotted line connecting y^6 to x^3. If one sets $y = v\sqrt{ax}$ and substitutes all the occurrences of y in the parallelogram, all the marked terms blocked off from the point corresponding to x^0y^0 by this line will be of a higher degree in x compared to the terms intercepted by the line. For instance, $xy^5 = v^5a^5x^{7/2}$ is of higher degree in x compared to $y^6 = v^6a^3x^3$. This procedure can be repeated, substituting all occurrences of x with $y^2/(v^2a)$. All the terms marked by asterisks that are not intercepted by the dotted line will be of higher degree in y. These terms can be discarded in the vicinity of the origin. One obtains the following fictitious equation, which approximates (7.19) near the origin:

$$y^6 - 7a^2x^2y^2 + 6a^3x^3 = 0. \tag{7.20}$$

This equation for $y = v\sqrt{ax}$ is reduced to

$$v^6 - 7v^2 + 6 = (v^2 - 1)(v^2 - 2)(v^2 + 3) = 0. \tag{7.21}$$

The real roots of (7.21) yield four representations for the curve in the vicinity of the origin:

$$y = \sqrt{ax} + \dots, y = -\sqrt{ax} + \dots, y = \sqrt{2ax} + \dots, y = -\sqrt{2ax} + \dots. \tag{7.22}$$

As Newton remarked: "[A]ny of these may be acceptable as an initial term ... depending on whether the decision is made to extract one or other of the roots."[44]

Say that the first approximation is $y \approx (ax)^{1/2}$. In order to find the other terms of the series, Newton developed a procedure of successive approximations inspired by a method that he had devised for approximating the roots of algebraic equations in one unknown (see figures 7.10 and 7.11 for a comparison of the two methods). Newton replaced $y = (ax)^{1/2} + p$ in the original equation (7.19), thus obtaining an equation in x and p.[45] By iterating the procedure with the analytical parallelogram, he obtained a new fictitious equation yielding an approximate solution for p in the vicinity of the origin. Step by step, a solution for y is obtained as a fractional power series in x which converges for x small.

It was natural for Newton to apply methods valid for polynomials to infinite series. Indeed, he called series infinite equations. Application of the method of

[44] MP, 3, p. 53. I note here that $y = \pm\sqrt{-3ax}$ approximates equation (7.19) near the origin for $x \leqslant 0$. Newton discarded imaginary roots of (7.21) because he sought a development of y as a fractional power series for positive values of x.

[45] Generally speaking, this equation will have fractional powers. If these cannot be eliminated by usual methods, a convention for placing the asterisks for fractional powers at intermediate points in Newton's parallelogram will be necessary.

$y^3 - 2y - 5 = 0$		$+2{,}10000000$ $-0{,}00544853$ $+2{,}09455147 = y$
$2 + p = y$	$+y^3$	$+8 + 12p + 6p^2 + p^3$
	$+2y$	$-4 - 2p$
	-5	-5
	Summa	$-1 + 10p + 6p^2 + p^3$
$0{,}1 + q = p$	$+p^3$	$+0{,}001 + 0{,}03q + 0{,}3q^2 + q^3$
	$+6p^2$	$+0{,}06 \quad + 1{,}2 \quad + 6{,}0$
	$+10p$	$+1, \quad + 10,$
	-1	$-1,$
	Summa	$+0{,}061 + 11{,}23q + 6{,}3q^2 + q^3$
$-0{,}0054 + r = q$	$+6{,}3q^2$	$+0{,}000183708 - 0{,}06804r + 6{,}3r^2$
	$+11{,}23q$	$-0{,}060642 \quad + 11{,}23$
	$+0{,}061$	$+0{,}061$
	Summa	$+0{,}000541708 + 11{,}16196r + 6{,}3r^2$
$-0{,}00004854 + s = r$		

Figure 7.10

Table for approximating the roots of an algebraic equation in one unknown (a) $y^3 - 2y - 5 = 0$. This table and the table shown in figure 7.11 were printed in the *De Analysi* (1669). Newton discussed these two methods as deeply interrelated. In the case of the algebraic equation (a), Newton began with $y = 2$ as an approximation for the root. Then $2 + p$ is substituted into (a) and the equation (b) $-1 + 10p + 6p^2 + p^3 = 0$ is obtained. Discarding nonlinear terms in p, this is reduced to the fictitious equation $-1 + 10p = 0$; thus, $p = 0.1$ is an approximate root for equation (b). Next, one sets $p = 0.1 + q$ and substitutes into (b), obtaining (c) $0.061 + 11.23q + 6.3q^2 + q^3 = 0$. Discarding nonlinear terms in q, this is reduced to the fictitious equation $0.061 + 11.23q = 0$; therefore $q = -0.0054$. Proceeding along similar lines, the value $r = -0.00004852 + s$ is obtained. By summing, Newton obtained the approximate value $2 + 0.1 - 0.0054 - 0.00004852$. Nowadays this method is presented using recurrence relations, which are geometrically interpreted by referring to the graph of $z = y^3 - 2y - 5$. Source: Newton, *Analysis per Quantitatum* (1711), p. 9. Courtesy of the Biblioteca Universitaria di Bologna.

resolution of affected equations to the arcsin series (7.13) yields a series for x in terms of z, namely,

$$x = z - z^3/6 + z^5/120 - z^7/5040 - \cdots . \tag{7.23}$$

This is, of course, the $\sin z$ series. Such a method of reversion of series, as Newton called it, allowed him to express the other trigonometric functions as well as the logarithm and the exponential. All these series were obtained and applied to a variety of quadrature problems in *De Analysi*.

I conclude this section by referring to the following concise explanation, which presents Newton's parallelogram method in modern form. In substituting the series $y = \sum_{k=0}^{\infty} b_k x^k$ for y in the polynomial equation $f(x, y) = \sum_{i,j} a_{i,j} x^i y^j = 0$, the

$y^3 + a^2y - 2a^3 + axy - x^3 = 0$ $y = a - \frac{x}{4} + \frac{x^2}{64a} - \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$ &c.		
$+a+p=y$	$+y^3$ $+a^2y$ $+axy$ $-2a^3$ $-x^3$	$+a^3+3a^2p+3ap^2+p^3$ $+a^3+a^2p$ $+a^2x+axp$ $-2a^3$ $-x^3$
$-\frac{1}{4}x+q=p$	$+p^3$ $+3ap^2$ $+4a^2p$ $+axp$ $+a^2x$ $-x^3$	$-\frac{1}{64}x^3+\frac{3}{16}x^2q-\frac{3}{4}xq^2+q^3$ $+\frac{3}{16}ax^2-\frac{3}{2}axq+3aq^2$ $-a^2x+4a^2q$ $-\frac{1}{4}ax^2+axq$ $+a^2x$ $-x^3$
$+\frac{x^2}{64a}+r=q$	$+3aq^2$ $+4a^2q$ $-\frac{1}{2}axq$ $+\frac{3}{16}x^2q$ $-\frac{1}{16}ax^2$ $-\frac{65}{64}x^3$	$+\frac{3x^4}{4096a}+\frac{3}{32}x^2r+3ar^2$ $+\frac{1}{16}ax^2+4a^2r$ $-\frac{1}{128}x^3-\frac{1}{2}axr$ $-\frac{3x^4}{1024a}+\frac{3}{16}x^2r$ $-\frac{1}{16}ax^2$ $-\frac{65}{64}x^3$
$+4a^2-\frac{1}{2}ax+\frac{9}{32}x^2)+\frac{131}{128}x^3-\frac{15x^4}{4096a}(+\frac{131x^3}{512a^2}+\frac{509x^4}{16384a^3}$		

Figure 7.11
Table for the resolution affected equations in *De Analysi* (1669). The example taken into consideration by Newton is (i) $y^3+a^2y-2a^3+axy-x^3=0$. First, when $x \approx 0$, the equation is reduced to a fictitious equation $y^3+a^2y-2a^3=0$, leading to a first approximation $y=a$. Then one substitutes $y=a+p$ into the original equation (i), and obtains an equation in x and p: (ii) $p^3+3ap^2+axp+4a^2p+a^2x-x^3=0$. Application of the analytical parallelogram arranged in terms of monomials x^mp^n allows one to reduce equation (ii) to a new fictitious equation $a^2x+4a^2p=0$, so that one gets an approximation $p=-\frac{1}{4}x$. Then one sets $p=-\frac{1}{4}x+q$ and substitutes it into (ii). By iterating this procedure, the series expansion, for $x \approx 0$, $y=a-x/4+x^2/(64a)+(131x^3)/(512a^2)+\cdots$ is obtained. Note a sign error $(-131x^3/512a^2)$ in the table. This mistake does not appear in Newton's manuscript. See MP, 2, p. 224. Source: Newton, *Analysis per Quantitatum* (1711), p. 11. Courtesy of the Biblioteca Universitaria di Bologna.

goal is to obtain 0; it is essential, therefore, that the terms of lower degree should cancel each other out. After substitution, the exponents of x are of the form $i+kj$. In order that the terms of lower degree might cancel each other out, there must be at least two of them; hence, at least two pairs (i_1, j_1) and (i_2, j_2) must exist in the expression $F(x,y)$, so that $i_1+kj_1=i_2+kj_2$. Newton's parallelogram serves the purpose of graphically identifying such terms.[46]

[46] This is how Newton's method is explained in Barbin, *A History of Algorithms* (1999), p. 194.

Such modern translations of Newton's method of resolution risk investing it with a power and generality that it lacked. Newton did not use a notation for functions or summations. The generality of the modern presentation of his method was not accessible to him. Most notably, the iterative, recursive character of Newton's approximation techniques, as Kollerstrom has emphasized in his study of the Newton-Raphson method, was spelled out only in presentations achieved by mathematicians of later generations.[47] This fact has two implications. First, Newton had to present his method by showing concrete examples of its application. He had to guide his student by illustrating the craft of problem solving, presenting variations of the examples taken into consideration and teaching how to overcome difficulties. As Cantor observed, "[T]he parallelogram [method] is just described, not proven."[48] Second, some of the difficulties that Newton experienced in printing his mathematics (see part VI) might be related to the fact that some of his methods were best suited for oral or scribal communication, like recipes that could be tested by means of persistent attempts.

What about the convergence of series? There are, indeed, several problems with Newton's method—such as the determination of the initial term and the convergence of the iterative procedure—that attracted the attention of mathematicians, including Joseph Raphson, Thomas Simpson, and Joseph-Louis Lagrange. I have not addressed this issue so far because Newton did not pay much attention to it. It is only much later that convergence became a central problem for mathematicians. In the 1680s, however, Newton developed a theory of limits that, at least to some extent, formalized his intuitive notions of convergence (chapter 9).[49]

7.6 New and Common Analyses

> I divine great enlargement of the bounds of the mathematical empire will ensue.
> (Oughtred to Robert Keylway, October 1645)[50]

In these lines Oughtred enthusiastically endorsed the idea that symbolic algebra, which he promoted in England, would lead to great improvements. The approach that Oughtred pioneered was continued in England by mathematicians such as John Pell, John Kersey, and John Collins.[51] The young Newton proudly saw himself as part of this tradition but went a step further by promoting a new analysis, which

---

[47] Kollerstrom, "Thomas Simpson and 'Newton's Method of Approximation'" (1992).

[48] "in der *Methodus Fluxionum* ist das Parallelogramm nur beschrieben, nicht bewiesen." Cantor, *Vorlesungen* (1901), p. 104.

[49] As mentioned, at the very end of *De Analysi*, Newton referred to *Elements*, X, 1, that is, to the basic proposition for the method of exhaustion, in order to develop an elementary, and too restrictive, convergence test. MP, 2, p. 245.

[50] Rigaud, *Correspondence of Scientific Men of the Seventeenth Century* (1841), 1, p. 65.

[51] Pycior, *Symbols, Impossible Numbers, and Geometric Entanglements* (1997), pp. 70–102.

surpassed the "vulgar" or common analysis typically found in works adopting the Cartesian method. While Descartes confined himself to finite equations, Newton intended to follow the lead of Wallis and Mercator and use *infinite* equations, viz., power series. It was this move that, Newton claimed, opened the doors to a new analysis, one much more powerful than the Cartesian. Still in 1676, in a letter addressed to Leibniz that summarized most of the results of *De Analysi*, Newton wrote,

> From all this it is to be seen how much the limits of analysis are enlarged by such infinite equations: in fact by their help analysis reaches, I might almost say, to all problems.[52]

In *De Analysi* Newton had noted,

> And whatever common analysis performs by equations made up of a finite number of terms (whenever it may be possible), this method may always perform by infinite equations: in consequence, I have never hesitated to bestow on it also the name of analysis.[53]

Later, Newton was to distance himself from such enthusiastic endorsement of the new analysis of the moderns (see chapter 9).

In his early works on series, such as *De Analysi*,[54] Newton made it clear that he was providing a "method for the resolution of problems,"[55] an analytical tool of discovery, and that he was addressing mathematicians whom he qualified as "analysts."[56] His method, Newton himself admitted, was "rather briefly explained than narrowly demonstrated."[57] This pragmatic approach is strongly reminiscent of Wallis's. Much like Wallis before him, Newton described his method as an analytical tool based on infinite procedures of summation; he explained it via successful problem solving rather than by rigorous demonstrations. The young Newton's endorsement of modern analytics is also evident in the opening lines of *De Methodis*:

> Observing that the majority of geometers, with an almost complete neglect of the ancients' synthetic method, now for the most part apply themselves to the cultivation of analysis and with its aid have overcome so many formidable difficulties that

[52] Newton to Oldenburg for Leibniz (June 13, 1676), *Correspondence*, 2, p. 39. "Ex his videre est quantum fines Analyseos per hujusmodi infinitas aequationes ampliantur: quippe quae earum beneficio, ad omnia, pene dixerim, problemata ... sese extendit." *Correspondence*, 2, p. 29.
[53] MP, 2, p. 241. "Et quicquid Vulgaris Analysis per aequationes ex finito terminorum numero constantes (quando id sit possibile) perficit, haec per aequationes infinitas semper perficiat: Ut nil dubitaverim nomen Analysis etiam huic tribuere." MP, 2, p. 240.
[54] And in the opening sections of *De Methodis*.
[55] For instance, Newton stated that his method provides an "analysin ad solutionem problematum." MP, 3, p. 35.
[56] MP, 2, p. 222.
[57] MP, 2, p. 207.

> they seem to have exhausted virtually everything apart from the squaring of curves and certain topics of like nature not yet fully elucidated: I found not amiss, for the satisfaction of learners, to draw up the following short tract in which I might at once widen the boundaries of the field of analysis and advance the doctrine of curves.[58]

Newton's method of discovery, based on series, was aimed at solving two related problems that had been avoided by Cartesian analysis: the squaring of curves and the study of mechanical curves.[59] The two problems, of course, are related, because the squaring of geometrical curves could define mechanical curves.[60] This was also the main topic of Wallis's *Arithmetica Infinitorum*. Like Wallis, Newton was deployed the "conformity between the operation in species and numbers."[61] As decimal numbers allow the approximation of roots of numerical equations, so infinite series allow approximation techniques for algebraic affected equations: a similarity that nobody, Newton observed with amazement, with the exception of Mercator had noted.[62]

Newton could obtain infinite series expansions by methods that were Wallisian in character. Indeed, he obtained his binomial series thanks to inductive and interpolation techniques. Further, Newton continued series at infinity relying on analogy; he stated that when a rule emerges after the formation of the first terms, this rule can be extended to higher-order terms without hesitation.[63]

The overall impression that Newton conveyed in *De Analysi* and in the opening sections of *De Methodis* was that he was delivering a new method of discovery addressed to those mathematicians, such as Collins and Wallis, able to appreciate advancements over common finite analysis, a method applied to the squaring of curves and based on analogies, Wallisian inductions, and interpolations. In *De Analysi* Newton showed little interest in framing his method in any systematic form; he rather proceeded on the basis of concrete examples. An isolated attempt to defend the foundation of such techniques can be found toward the end of *De Analysi*:

> [D]eductions in [this method, [a]to be judged analytical[a]] are not less certain than in the other [common analysis], nor its equations less exact, even though we, mere

[58] MP, 3, p. 33. Translation by Whiteside.
[59] See also MP, 2, pp. 207, 233.
[60] For instance, the squaring of the hyperbola yields a logarithmic relation. Newton referred to the fact that the squaring of geometrical curves yields mechanical curves by referring to the latter as "quantities which cannot be determined and expressed by any geometrical technique, such as the areas and length of curves." MP, 3, p. 79.
[61] MP, 2, p. 213.
[62] MP, 3, p. 33.
[63] "After the roots [of affected equations] have been extracted to a suitable period, they may sometimes be extended at pleasure by observing the analogy of the series." MP, 3, p. 61. Cfr. MP, 2, p. 237.

> men possessed only with finite intelligence, can neither designate all their terms nor so grasp them as to ascertain exactly the quantities we desire from them.[64]

Newton defended infinite series as something certain, against Descartes' finitist standards of certainty, on the grounds that only the limits of human intelligence do not allow us to grasp all the series' terms. Here, too, Newton was behaving like a typical practitioner of the new analysis. His was a defensive strategy frequently adopted by the seventeenth-century promoters of infinite products and series.

Infinite fractional power series (whether obtained by binomial expansions, long divisions, root extractions, or the more elaborate iterative techniques for the resolution of affected equations) allowed Newton to deal with mechanical curves and quadrature problems with incredible ease. The quadrature of the cycloid or quadratrix are two examples of this.[65] Quadratures were carried out by what in modern terms would be called term-wise integration. Newton, however, soon realized that a much more powerful approach to quadrature was provided by the fundamental relation between area-problems and tangent-problems (nowadays identified as the fundamental theorem of the calculus). Newton developed this approach in the October 1666 tract on fluxions[66] and systematized it in detail in *De Methodis*, which we turn to in the following chapter.

---

[64] MP, 2, pp. 240–3. ^{aa} = "Conclusio quod haec methodus Analytica censenda est" added in margin. MP, 2, p. 240. Translation by Whiteside.

[65] "If the curve is mechanical it yet by no means spurns our method." MP, 2, p. 239. Translation by Whiteside.

[66] Add. 3958.3, ff. 48v–63v in MP, 1, pp. 400–48.

8 The Analytical Method of Fluxions

> The chief Principle, upon which the Method of Fluxions is here built, is this very simple one, taken from Rational Mechanicks; which is, That Mathematical Quantity, particularly Extension, may be conceived as generated by continued local Motion; and that all Quantities whatever, at least by analogy and accommodation, may be conceived as generated after a like manner.
>
> — John Colson, 1736

8.1 Barrow

8.1.1 Barrow as Newton's Mentor

Newton's first attempts to codify the method of fluxions date from the October 1666 tract.[1] This chapter, however, deals with the analytical version that Newton fully developed in *De Methodis* (1671) and in *De Quadratura* (1691–1692).[2] The method is divided into a direct and an inverse part. Newton considered the techniques of the direct method as having been brought to perfection in his 1671 treatise. After 1671 he sought both to improve the algorithm of the inverse method and reach a better conceptual foundation for the direct method. Newton kept on working on these issues until the early 1690s, when he composed *De Quadratura*, a work that provides the most advanced refinement of the method of fluxions.

According to Newton's standards, analysis had to be followed by synthesis. Therefore, from the early 1670s he attempted to develop a synthetic method of fluxions (see chapter 9).

While the method of series was developed by Newton by deriving inspiration from Wallis's work, in the method of fluxions he followed the steps of Barrow, even though the influence of Barrow is less manifest. Thus, some features of Barrow's work that most probably were important for Newton are considered first.

In 1663, when Newton was a young student in Cambridge, Isaac Barrow, a theologian and a mathematician highly esteemed by his contemporaries, was appointed

---

Epigraph from the Preface by John Colson to Newton, *The Method of Fluxions and Infinite Series* (1736), p. xi.

[1] Add. 3958.3, ff. 48v–63v in MP, 1, pp. 400–48 (see chapter 1).

[2] See Abbreviations and Conventions for bibliographical details on *De Methodis* and *De Quadratura*.

to the newly instituted Lucasian Chair of Mathematics.[3] The value and nature of Barrow's mathematical research has been the object of much debate since the inception of the Newton-Leibniz controversy. It is evident that some of his results are related to what is identified in the literature as the infinitesimal calculus, a term that neither Barrow nor Newton ever used. The extent to which all this can be taken as proof of Barrow's contribution to the calculus, however, is unclear. Barrow proved some geometrical results concerning the drawing of tangents and the squaring of curves that were later identified as equivalent to the so-called fundamental theorem of the calculus, that is, as the statement of the inverse relation between differentiation and integration.

In 1916, Child defended the thesis according to which Barrow was the first inventor of the calculus:

> Barrow was the first inventor of the Infinitesimal Calculus; Newton got the main idea of it from Barrow by personal communication; and Leibniz also was in some measure indebted to Barrow's work, obtaining confirmation of his own original ideas, and suggestions for their further development, from the copy of Barrow's *Lectiones Geometricae* that he purchased in 1673.[4]

Child's claim, however, cannot withstand the careful reconstruction of Newton's and Leibniz's independent paths to discovery—respectively, of the method of series and fluxions and of the differential and integral calculus—which most notably emerges in the seminal studies of Hall, Hofmann, and Whiteside.[5] It is now clear that neither Newton nor Leibniz was exclusively indebted to Barrow, since both drew from a large body of mathematical literature; besides, their original contributions proved momentous.

Even though the question of Barrow's priority often proves misleading, it is difficult to deny that Barrow's presence in Cambridge must have shaped Newton's mathematical ideas to some extent. As Feingold has shown, Barrow and Newton might have been in contact to exchange ideas on mathematics during Newton's formative years. In his retrospective memoranda Newton always attributed a major role to Barrow as his mathematical mentor. Further, it is through Barrow that Newton got in touch with John Collins in London; the Lucasian Chair was conferred to Newton thanks to Barrow's recommendation; and in 1670, Barrow asked Newton to edit his *Lectiones Geometricae*.[6] All these factors substantiate the hypothesis that

---

[3] For an intellectual biography of Barrow, see Feingold, "Isaac Barrow: Divine, Scholar, Mathematician" (1990).

[4] From Child's Preface to Barrow, *Geometrical Lectures* (1916), p. vii.

[5] Hofmann, *Leibniz in Paris* (1974); Hall, *Philosophers at War* (1980); Whiteside's commentary in MP, 1 and 8.

[6] Feingold, "Newton, Leibniz, and Barrow Too" (1993). I do not have space here to present the details of Feingold's argument. It seems to me that he has convincingly shown untenable the

between Barrow and Newton a close relationship existed at some time. Arthur has pointed to notable similarities in language between Newton and Barrow on themes such as absolute time and space, and the generation of magnitudes by motion.[7] After all, Newton himself, in 1712, recognized that "it is from him [Barrow] that I had the language of momenta & incrementa momentanea & this language I have always used & still use."[8] It is not my purpose to explore the matter of Barrow's early influence on Newton and Newton's indebtedness to Barrow any further. In the years 1664–1669, Barrow and Newton lived close to one another; if the two men were ever in contact with one another (as seems most likely), they did not leave any traces for future historians. Certainly, from 1669 onward Newton became deeply acquainted with Barrow and his mathematical work: from this date at least, Newton's own work shows signs of Barrow's influence. My aim in the following pages is to highlight certain aspects of Barrow's mathematics that bear some resemblance to Newton's early method of fluxions as expounded in *De Methodis*.[9]

8.1.2 Generation of Magnitudes by Motion

Both *Lectiones Mathematicae* and *Lectiones Geometricae*, which Barrow delivered during his tenure at Cambridge from 1663 to 1669, clearly state that the object of mathematics is geometrical magnitudes generated by motion.[10] This was to become a basic tenet of Newton's fluxional method as well. The continuous motion of a point generates a line, the motion of a line generates a surface, and the motion of a surface generates a solid.[11]

Conceiving objects as generated by continuous motion presented two advantages that Newton appreciated. The first is that the limiting procedures deployed in calculating tangents and areas can be grounded on the continuity of motion, that is, it is possible to claim that the limits determined by such procedures exist because of the continuity of the generating motion. Further, the continuity observed in physical motions allows mathematics to be envisaged as a language applicable to the study of the natural world.

---

thesis that Newton could not derive any idea from a Professor of Mathematics who belonged to his College and who later did everything he could in order to secure Newton's academic position.

[7] Arthur, "Newton's fluxions and equably flowing time" (1995). See also Whiteside in MP, 3, pp. 70–2.

[8] *Correspondence*, 5, p. 213.

[9] For a detailed analysis of Barrow's mathematics, see Mahoney, "Barrow's mathematics" (1990). See also Malet, "Barrow, Wallis, and the Remaking of Seventeenth Century Indivisibles" (1997); Pycior, "Mathematics and Philosophy" (1987); Sasaki, "The Acceptance of the Theory of Proportion" (1985).

[10] Barrow, *Lectiones Geometricae* (1670), *Lectiones Mathematicae* (1683). Both reprinted in Barrow, *Mathematical Works* (1860).

[11] Barrow, *Mathematical Works* (1860), p. 188.

In this context, Barrow, proving himself an innovator rather than a conservative, did away in *Lectiones Mathematicae* with the traditional distinction between pure and mixed (or concrete) mathematics, by stating that since continuous magnitude is the affection of all things, there is no part of "physics" that is not reducible to geometry. In conclusion Barrow went so far as to claim that "Mathematics is ... co-extended with physics,"[12] a statement that paved the way for the legitimation of Newton's "geometrical philosophers" (see chapter 2). By stating the generality of his kinematic geometry, Barrow broke both with the limitations of the Cartesian canon and with the traditional Aristotelian disciplinary taxonomies.[13]

8.1.3 Symptomata of Curves Deduced from their Generation

Barrow's aim in *Lectiones Geometricae* was to "study and display the affections of curves which emerge from the composition of motions."[14] In the first lectures Barrow explained in detail how curves can be generated by composition of motions in many different ways. One could think, for instance, of a point sliding along a line which itself has a translational or a rotational motion (think about the generation of a Galilean parabola through the composition of uniform and accelerated motion, or the generation of an Archimedean spiral through the composition of uniform rotation and uniform rectilinear motion in (§1.3)). Another way of generating curves is by the concurrent motion of two lines so that their intersection traces a curve (think about the generation of the quadratrix in (§1.3)). As Mahoney observed about Barrow's mathematics, "[T]he properties of concern to Barrow follow directly from the curves' mode of generation."[15] Most notably, in the sixth lecture Barrow studied how the subtangent to a curve generated by motion could be determined "without the trouble or wearisomeness of calculation" in function of the generating motions.[16] In 1665, Newton devised a kinematic method for determining tangents

---

[12] "those which are called mixed or concrete mathematical sciences, are rather so many examples only of Geometry, than so many sciences separate from it: for once they are disrobed of particular Circumstances, and their own fundamental and principal Hypotheses come to be admitted (whether sustained by a probable Reason, or assumed gratis) they become purely Geometrical." "For magnitude is the common affection of all physical things, it is interwoven in the Nature of Bodies, blended with all corporeal Accidents." "I say there is no part of this [Physics] which does not imply Quantity ... and consequently which is not in some way dependent on Geometry." "Mathematics ... is adequate and co-extended with physics." Barrow, *The Usefulness of Mathematical Learning* (1734), pp. 27, 21, 22, 26.

[13] Hill, "Neither Ancient nor Modern" (1996).

[14] "Propositum est nobis e compositione motuum ... emergentes linearum affectiones indagare ac exponere." Barrow, *Lectiones Geometricae* (1670), p. 29 = *Mathematical Works* (1860), p. 191.

[15] Mahoney, "Barrow's Mathematics" (1990), p. 207.

[16] Barrow, *Mathematical Works* (1860), p. 208. Translation by Mahoney in "Barrow's Mathematics" (1990), p. 214.

to "mechanicall lines" (§1.3). It is therefore possible to discern Barrow's influence here.

8.1.4 Subtangents *ex calculo*

Barrow also had a method for finding subtangents *ex calculo*.[17] He discussed it at the close of Lecture 10 following the advice of a "friend": the young Newton, who was then rather enthusiastic about algorithmic methods. Barrow's method is based on the limitative assumption that there is an equation relating the abscissa and the ordinate of the curve. Barrow proceeded as follows (see figure 8.1):

> Let AP, PM be straight lines given in position (of which PM cuts the proposed curve at M), and suppose MT to be tangent to the curve at M and to cut the line AP at T. Now, to find out the quantity of this line PT, I posit the arc MN as indefinitely small. Then I draw lines NQ parallel to MP and NR [parallel] to AP. I call $MP = m$, $PT = t$, $MR = a$, $NR = e$; the remaining lines determined by the special nature of the curve and useful to the proposition I designate by names. But MR and NR (and by means of them MP and PT) I compare to one another by an *equation* expressed in terms of calculation [ex calculo]. In doing so I observe these rules:
>
> 1. In the computation I reject all terms in which a power of a or e occurs or in which these are multiplied by one another (because these terms will count for nothing).

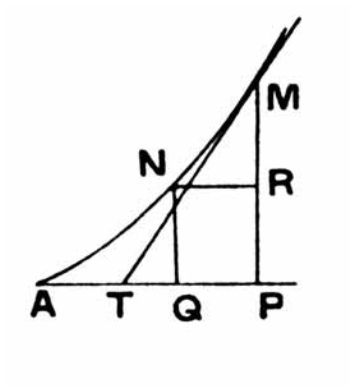


Figure 8.1
Diagram for Barrow's method of tangents. Source: Barrow, *Geometrical Lectures* (1916), p. 120. Courtesy of the Biblioteca Angelo Mai (Bergamo).

[17] The subtangent is defined as the segment of the x-axis lying between the x-coordinate of the point at which a tangent is drawn to a curve and the intercept of the tangent with the x-axis (see note 52).

2. After the equation has been set up, I reject all terms consisting of letters designating known or determinate quantities, or in which a or e does not occur (because these terms, when brought to one side of the equation, will equal nothing).
3. I substitute m (or MP) for a, and t (or PT) for e. From this finally the quantity PT [the subtangent] itself is determined.

> And if an indefinitely small part of any curve should enter the calculation, substitute in its place a suitably chosen small part of the tangent, or any line equivalent to it (by virtue of the indefinite smallness of the curve).[18]

As Mahoney noted, there is nothing radically new in this method. Pierre de Fermat and many others after him had deployed similar techniques to calculate subtangents. Barrow's technique depended upon the assumption that an equation relating the abscissa and the ordinate is available. Newton adopted techniques for drawing tangents that resemble both Barrow's kinematic and algorithmic methods. What remained unclear in Barrow's formulation was how to deal with equations in which radicals occur and, more important, how to deal with mechanical curves generally. There is no doubt that Barrow found these two cases burdened by "wearisome calculation" and that he gave preference to the kinematic method, similar to the one already developed by Gilles Personne de Roberval, over the algorithmic one precisely because it was unclear how the latter might be extended to nonalgebraic curves.[19]

8.1.5 Problem Reduction

Another aspect of *Lectiones Geometricae* that shows resemblances with Newton's fluxional method is the fact that Barrow organized his work around two related problems: the finding of tangents and the finding of curvilinear areas. At the opening of Lecture 6, Barrow stated that he would pursue two goals: "the finding of tangents without the trouble or wearisomeness of calculation" and the "ready determination of the dimension of many magnitudes by the help of tangents which have been drawn."[20] Here Barrow showed clear awareness of the fact that the determination of the dimensions (i.e., the areas and volumes) of curvilinear figures can be achieved thanks to theorems that concern the finding of tangents. A fundamental relation between two apparently disconnected problems was thus identified

[18] Barrow, *Mathematical Works* (1860), pp. 246–7. Translation by Mahoney in "Barrow's Mathematics" (1990), pp. 225–6.

[19] Barrow, however, was able to calculate the subtangent to the quadratrix and to some trigonometric curves. See the close of Lecture 10.

[20] "Versantur autem praecipue quae proferemus, partim *circa tangentium absque calculi molestiam vel fastidio investigationem simul ac demonstrationem expeditam . . . partim circa multarum magnitudinum dimensiones, tangentium designatarum ope, quam promptissime determinandas.*" Barrow, *Mathematical Works* (1860), p. 209.

in *Lectiones Geometricae*. This lesson is likely to have polarized the attention of Barrow's gifted student, who broached these two problems in the direct and the inverse method of fluxions, respectively.

8.1.6 Transmutation of Areas

How tangents and quadrature problems are related is explained by Barrow in a number of propositions. One can cite Proposition 11 from Lecture 10 (figure 8.2) and Proposition 19 from Lecture 11 (figure 8.3):

> **Proposition 11, Lecture 10**. Let ZGE be any curve [see figure 8.2] of which the axis is AD; and let ordinates applied to this axis, AZ, PG, DE, continually increase from the initial ordinate AZ; and also let AIF be a line such that, if any straight line EDF is drawn perpendicular to AD, cutting the curves in the points E, F, and AD in D, the rectangle contained by DF and a given length R is equal to the intercepted space $ADEZ$; also let $DE : DF = R = OT$, and join DT. Then TF will touch the curve AIF.
>
> For, if any point I is taken in the line AIF (first on the side of F towards A), and if through it IG is drawn parallel to AZ, and KL is parallel to AD, cutting the given line as shown in the figure; then
>
> $$LF : LK = DF : DT = DE : R,$$
>
> or
>
> $$R \times LF = LK \times DE.$$

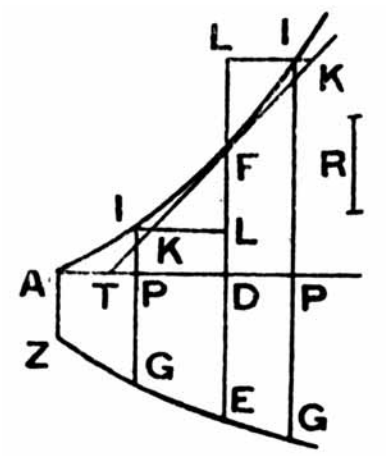


Figure 8.2
Diagram for Proposition 11 from Lecture 10. Source: Barrow, *Geometrical Lectures* (1916), p. 117. Courtesy of the Biblioteca Angelo Mai (Bergamo).

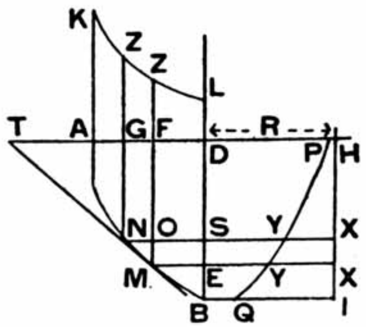


Figure 8.3
Diagrams for Proposition 19 from Lecture 11. Source: Barrow, *Geometrical Lectures* (1916), p. 135. Courtesy of the Biblioteca Angelo Mai (Bergamo).

But, from the stated nature of the lines DF, PK, we have $R \times LF =$ area $PDEG$; therefore $LK \times DE =$ area $PDEG < PD \times DE$; hence $LK < DP < LI$.

Again, if the point I is taken on the other side of F, and the same construction is made as before, plainly it can be easily shown that $LK > DP > LI$.

From which it it is quite clear that the whole of the line $TKFK$ lies within or below the curve $AIFI$.

Other things remaining the same, if the ordinates, AZ, PG, DE continually decrease, the same conclusion is attained by similar argument.

Proposition 19, Lecture 11. Again, let AMB [see figure 8.3] be a curve of which the axis is AD and the BD be perpendicular to AD; also let KZL be another line such that, when any point M is taken in the curve AB, and through it are drawn MT a tangent to the curve AB, and MFZ parallel to DB, cutting KZ in Z and AD in F, and R is a line of given length, $TF : FM = R : FZ$. Then the space $ADLK$ is equal to the rectangle contained by R and DB.

For, if $DH = R$ and the rectangle $BDHI$ is completed, and MN is taken to be an indefinitely small arc of the curve AB, and MEX, NOS are drawn parallel to AD; then we have

$$NO : MO = TF : FM = R : FZ;$$

therefore $NO \times FZ = MO \times R =$, and $FG \times FZ = ES \times EX$.

Hence, since the sum of such rectangles as $FG \times FZ$ differs only in the least degree from the space $ADLK$, and the rectangles $ES \times EX$ from the rectangle $DHIB$, the theorem is quite obvious.[21]

[21] Child's translation in Barrow, *Geometrical Lectures* (1916), pp. 116–9, 135.

These propositions can be immediately understood (perhaps too optimistically) as a statement of the fundamental theorem of the calculus. Child's claims about Barrow's priority over Leibniz and Newton are mainly based on these propositions. I do not wish to enter this discussion here, but I refer the reader to Mahoney's study, which concluded that "what in substance becomes part of the fundamental theorem of the calculus is clearly not fundamental for Barrow."[22] Indeed, as Mahoney observed, Barrow did not give particular emphasis to these two propositions and did not relate them to one another; they occur in two separate lectures and seem to play independent roles. Further, Barrow did not translate these propositions into an algorithm for determining areas in function of antiderivatives. There is much wisdom and historical sensitivity in these cautionary remarks. But I fear that in subjecting Barrow's *Lectiones Geometricae* to evaluations that are polarized by the purpose of refuting Child's wild claims one risks failing to understand his mathematics in its own terms.

Barrow was not interested in developing an algorithm for broaching the problems concerning curves to which he devoted *Lectiones Geometricae*, since he was convinced of the superior generality of geometry over algebra, an altogether justified position given the fact that algebraic techniques for dealing with mechanical curves had yet to be invented by his young protégé. What Barrow wished to do, it seems to me, was to demonstrate general relations between propositions concerning curvilinear areas and propositions concerning tangents. Such relations had already appeared in the literature on specific curves. For Barrow, geometry was the means to prove that such relations are quite general and hold for any curve independently of its algebraic representability. Geometry offered Barrow a language appropriate for expressing general theorems concerning the tangents and areas of curves.

Whatever Barrow's awareness of the centrality of his theorems on areas and tangents, it is a fact that in 1665 Newton based his first demonstration of the inverse relation between area-problems and tangent-problems on a proposition that is strikingly similar to Barrow's Proposition 11 from Lecture 10 (§8.2.6 and figure 8.7).[23] Further, in 1670 he turned to Barrow's theorems on quadratures (as those surrounding Proposition 19 from Lecture 11) to find a synthetic construction of his analytical algebraic method of quadratures (§9.1).

8.1.7 Apagogical Proofs

Barrow employed infinitesimal magnitudes in his proofs. He often stated that such proofs could be reframed by means of more "apagogical" procedures. The term *apagogical* was used in scholastic logic to designate a reasoning that demonstrates a

[22] Mahoney, "Barrow's Mathematics" (1990), p. 236.
[23] MP, 1, pp. 302–5, 313–5.

proposition A by proving the impossibility of the negation of A. These *ad absurdum* proofs were generally lengthy, and Barrow claimed that it was only for the sake of brevity and perspicuity that he used less rigorous proofs where infinitesimals occurred. In the second appendix to Lecture 12 of *Lectiones Geometricae*, he wrote,

> Having regard for brevity and perspicuity (mainly the latter), the preceding results were proven by direct arguments, by which not only the truth is cogently enough confirmed, but also their origins most neatly appear. But for fear anyone less used to this sort of arguments had difficulty, we shall add the following short notes. With them the said arguments are secured and with their help apagogical proofs of the preceding results will be easily worked out.[24]

In the appendix Barrow developed an apagogical demonstration of the assumption that a curvilinear area can be equated with the summation of an infinite number of areas of infinitesimal parallelograms. He did so by means of *ad absurdum* reductions reminiscent of the Archimedean method of exhaustion. Barrow considered the curvilinear surface shown in figure 8.4 and conceived it as being subdivided into an infinity of parallelograms, whose bases ZZ are infinitesimal.

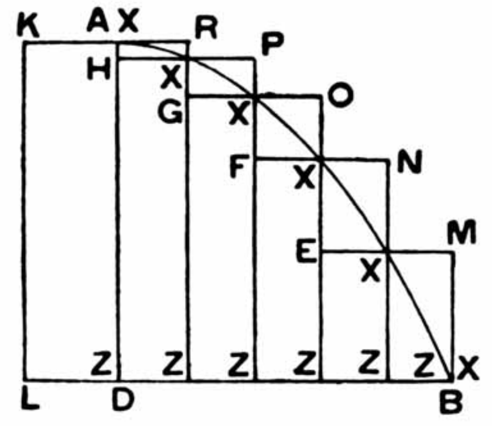


Figure 8.4
Diagram for Barrow's apagogical proof. Source: Barrow, *Geometrical Lectures* (1916), p. 172. Courtesy of the Biblioteca Angelo Mai (Bergamo).

[24] "Brevitati simul ac perspicuitati (huic autem praecipue) consulentes recto discursu comprobata dedimus; quali non modo veritas, opinor, satis firmatur, at ejusdem origo limpidius apparet. Verum ne quis, minus hujusmodi ratiociniis adsuetus, haereat, ista paucula subdemus, quibus tales discursus communiantur, quorumque subsidio non difficile conficiantur *Propositorum demonstrationes apagogicae*." Barrow, *Mathematical Works* (1860), p. 284. Translation in Malet, *From Indivisibles to Infinitesimals* (1996), p. 49.

Barrow aimed to prove that the assumption that rectilinear circumscribed and inscribed figures are greater or smaller than the curvilinear surface leads to contradiction. The gist of his argument consists in showing that their difference (which is equal to rectangle $ADLK$) is infinitesimal, that is, its area is "less than any given magnitude."

As Malet observed in his careful study on infinitesimal techniques in the seventeenth century, Barrow

> was not concerned about the use of infinitesimals and did not make an attempt to get rid of them. What did concern him was to show that the difference between an aggregate of infinitesimals each one being not truly identical with a part of the whole surface, and the surface is less than any finite magnitude.[25]

Newton also offered proofs based on infinitesimals "for the sake of brevity" and attempted a more rigorous foundation of them along the lines of exhaustion techniques (see chapter 9). While Newton's approach to similar demonstrations was different from Barrow's (based as it was on limiting procedures), he retained Barrow's diagram (see figure 8.4) and the idea that the difference between the circumscribed and inscribed figures is equal to the rectangle $ADLK$ (see figure 9.3).

8.2 Preliminaries to the Method of Fluxions

8.2.1 An Analytical Art

After this brief introduction to Barrow's *Lectiones Geometricae*, I now turn to Newton's *De Methodis Serierum et Fluxionum*. Several features of Barrow's *Lectiones* surface in Newton's method of series and fluxions.

While *De Analysi* (see chapter 7) is a short tract mainly devoted to series expansions and their use in quadratures, *De Methodis* (written in 1670–1671) is a long treatise whose aim is to deliver the applications of an "analytical art" useful to the study of the "nature of curves." In this masterpiece Newton systematized his early work on tangents and quadratures by reworking and greatly extending the October 1666 tract on fluxions.

Newton's method of fluxions is deeply intertwined with his method of series. Indeed, in the opening lines of *De Methodis* Newton incorporated and expanded *De Analysi* by presenting his methods of series via long division, root extraction, and the resolution of affected equations (§7.5). Then he wrote,

> So much for computational methods of which in the sequel I shall make frequent use. It now remains, in illustration of this analytical art, to deliver some typical problems and such especially as the nature of curves will present.[26]

---

[25] Malet, *From Indivisibles to Infinitesimals* (1996), p. 49.

[26] MP, 3, p. 71. "Hactenus de modis computandi quorum post hac frequens erit usus. Jam

This statement defines Newton's object of study and method: *De Methodis* is a work on analysis applied to the resolution of problems concerning curves. Synthesis, with its constructions and demonstrations is given little space. Newton, however, retained the structure of the analytical/synthetic canon. He always began from analysis: he considered a geometrical problem, translated it into algebra, and manipulated symbols until he achieved a resolution of the problem. He then briefly turned to synthesis, that is, he provided a geometrical construction and a geometrical demonstration that this construction is what is required in order to reach a solution of the problem considered. The last stage, however, which was Barrow's main concern in *Lectiones Geometricae*, is only briefly touched upon. Newton developed the synthesis more fully in an Addendum composed in 1671 and in a treatise entitled "Geometria Curvilinea," written around 1680 (see chapter 9).

8.2.2 Basic Definitions

In *De Methodis*, Newton made it clear that the objects to which his analytical method applied are geometrical quantities generated by a process of flow in time. For instance, the motion of a point generates a line, and the motion of a line generates a surface:

1. The quantities generated by flow are called fluents.
2. Their instantaneous speeds are called fluxions.
3. The moments of the fluent quantities are "the infinitely small additions by which those quantities increase during each infinitely small interval of time."[27]

Therefore, consider a point that flows with variable speed along a straight line (figure 8.5). The distance covered at time t is the fluent, and the instantaneous speed is the fluxion. The indefinitely or infinitely small parts by means of which the fluent increases after indefinitely or infinitely small intervals of time are the moments of the fluent quantity.[28] Newton further observed that the moments are "as the speed of flow" (i.e., the fluxions).[29] His reasoning was based on the idea that during an infinitely small period of time the fluxion remains constant, and hence the moment is proportional to the fluxion.

restat ut in illustrationem hujus Artis Analiticae tradam aliquot Problematum specimina qualia praesertim natura curvarum ministrabit." MP, 3, p. 70.

[27] MP, 3, p. 81. "additamenta infinite parva quibus illae quantitates per singula temporis infinite parva intervalla augentur." MP, 3, p. 80.

[28] See the definitions given at MP, 3, pp. 78, 80. Newton referred to infinitesimal increments employing the term infinitely as well as indefinitely.

[29] MP, 3, p. 79. "sunt ut fluendi celeritates." MP, 3, p. 78.

Figure 8.5
Relations between fluent, fluxion, and moment. ©Niccolò Guicciardini.

Newton warned the reader not to identify the time of the fluxional method with real time. Any fluent quantity whose fluxion is assumed constant ($\dot{x} = 1$) plays the role of fluxional time ($x = t + C$). The choice of x as the time variable is arbitrary (and further if $y = kx$ also $\dot{y}$ is constant) and in general dictated by computational convenience. His language here is reminiscent of Barrow's *Lectiones Geometricae* (§8.1.2).[30]

8.2.3 Notation

Contrary to Barrow, Newton developed in *De Methodis* an algorithmic approach extensively. His notation, however, is not particularly handy. For instance, he might employ a, b, c, d for constants, v, x, y, z for fluents, and l, m, n, r for the respective fluxions, so that, for instance, m is the fluxion of x, and n is the fluxion of y, etc. The indefinitely (or infinitely) small interval of time is always denoted by o, so that the moment of y is no. This notation is very confusing; in what follows, I always employ a notation that Newton invented much later.

In fact, it was only in the 1690s that Newton introduced the now standard notation according to which the fluxion of x is denoted by $\dot{x}$, and the moment of x by $\dot{x}o$. Fluxions themselves can be considered fluent quantities. In the 1690s, Newton denoted the second fluxion of x with $\ddot{x}$ (whereas he had previously employed letters, so that, for instance, q was the second fluxion of y). Multiple points or numbers placed over the fluent symbols can denote higher-order fluxions.

Newton did not use a consistent notation for the area of the surface under a curve.

[30] Cfr., for instance: "To every instant of time, or to every indefinitely small particle of time; (I say 'instant' or 'indefinite particle' because, just as it matters nothing at all whether we understand a line to be composed of innumerable points or of indefinitely small linelets [*lineolae*], so it is all the same whether we suppose time to be composed of instants or of innumerable minute timelets [*tempusculis*]; at least for the sake of brevity we shall not fear to use instants in place of times however small, or points in place of the linelets representing timelets); to each moment of time, I say, there corresponds some degree of velocity which the moving body should be thought to have then; to that degree corresponds some length of space traversed (for here we consider the moving body as a point and thus the space only as length)." From Lecture 1 of *Lectiones Geometricae* (1670). See Barrow, *Mathematical Works* (1860), pp. 167–8. Translation by Mahoney in "The Mathematical Realm of Nature" (1998), p. 743.

Most often, he used words such as "the area of" or (as recorded in one instance) a capital Q before the analytical expression of the curve.[31] In some cases he used "$\boxed{a/x^2}$" for "the area of the surface under the curve of equation $y = a/x^2$."[32] In the 1690s Newton also employed $\acute{x}$ to denote a fluent quantity whose fluxion is x. The limits of integration were either evident from the context or explained in words, not symbols.

8.2.4 Problem Reduction

In the *De Methodis*, Newton applied the method of series and fluxions to several problems. The main ones were how to find maxima and minima of varying magnitudes, how to determine tangents and curvatures of plane curves, and how to calculate curvilinear areas and arclengths. Thanks to the representation of quantities as generated by a continuous flow, all these problems can be reduced to the following two:

> Problem 1. Given the length of the space continuously (that is, at every time), to find the speed of motion at any time proposed.
>
> Problem 2. Given the speed of motion continuously, to find the length of the space described at any time proposed.[33]

The problems of finding tangents, maxima and minima, and curvatures are related to Problem 1, the problems of finding curvilinear areas and arc lengths are related to Problem 2.

8.2.5 The Role of the Fundamental Theorem

Newton demonstrated the fundamental theorem in some of his early manuscripts and in *De Analysi*. Nowadays the fundamental theorem is understood as a statement that the two central operations of calculus, differentiation and integration, are inverse operations. An important consequence of this is the possibility of computing integrals by using an antiderivative of the function to be integrated, a consequence that Newton fully appreciated and deployed in *De Methodis* (§8.4.3).

The fact that the calculation of curvilinear areas (in Newton's terms, the problem of quadrature) can be reduced to Problem 2 is implied by Newton's proof of the fundamental theorem. Newton demonstrated that if z is the area generated by the

---

[31] Notes to the *De Quadratura* (early 1690s). Add. 3960.8, f. 155. MP, 7, p. 156.

[32] In Leibnizian terms, one would have $\int(a/x^2)dx$.

[33] MP, 3, p. 71. "1. Spatij longitudine continuo (sive ad omne tempus) data, celeritatem motus ad tempus propositum invenire. 2. Celeritate motus continuo data longitudinem descripti spatij ad tempus propositum invenire." MP, 3, p. 70.

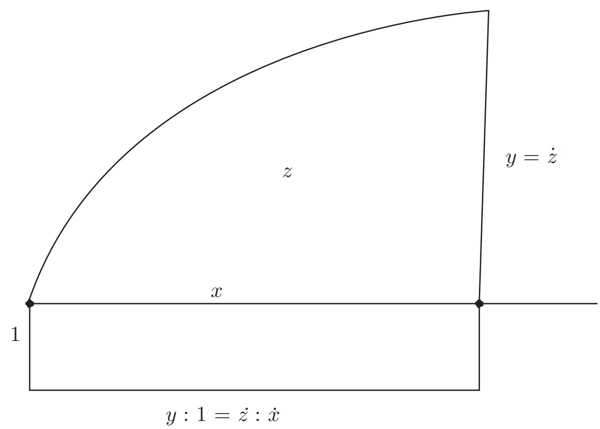


Figure 8.6
Relations between area z, abscissa x, and ordinate y. ©Niccolò Guicciardini.

continuous uniform flow ($\dot{x} = 1$) of ordinate y, then $y = \dot{z}$ (figure 8.6). Note that the conception of quantities as generated by continuous flow allowed Newton to conceive the problem of determining the area under a curve as a special case of Problem 2.

The reduction of arclength problems to Problem 2 depends on the application of the Pythagorean theorem to the triangle formed by the moment of arclength s, the moment of the abscissa x, and the moment of the ordinate y: $\dot{s}o = \sqrt{(\dot{x}o)^2 + (\dot{y}o)^2}$. Therefore, $s = \boxed{\sqrt{\dot{x}^2 + \dot{y}^2}}$.

8.2.6 Proofs of the Fundamental Theorem

Proof (1665) Newton discovered the fundamental theorem in 1665.[34] His reasoning, which strongly resembles Barrow's Proposition 19 from Lecture 11 (§8.1.6), refers to two particular curves $z = x^3/a$ and $y = 3x^2/a$. However, it is completely general; the only property that matters is that y be equal to the slope of z (see figure 8.7, where z is drawn above the x-axis, and y is drawn below the x-axis).[35] More precisely, y is defined as

$$bg = dh\frac{\beta m}{\Omega\beta}, \tag{8.1}$$

[34] Add. 4000, ff. 120r–133v. "A Method Whereby to Square Those Crooked Lines Wch May Be Squared." MP, 1, pp. 302–13.
[35] In a second draft Newton employed $z = a^3/x$ and $y = -a^3/x^2$. MP, 1, pp. 314–21.

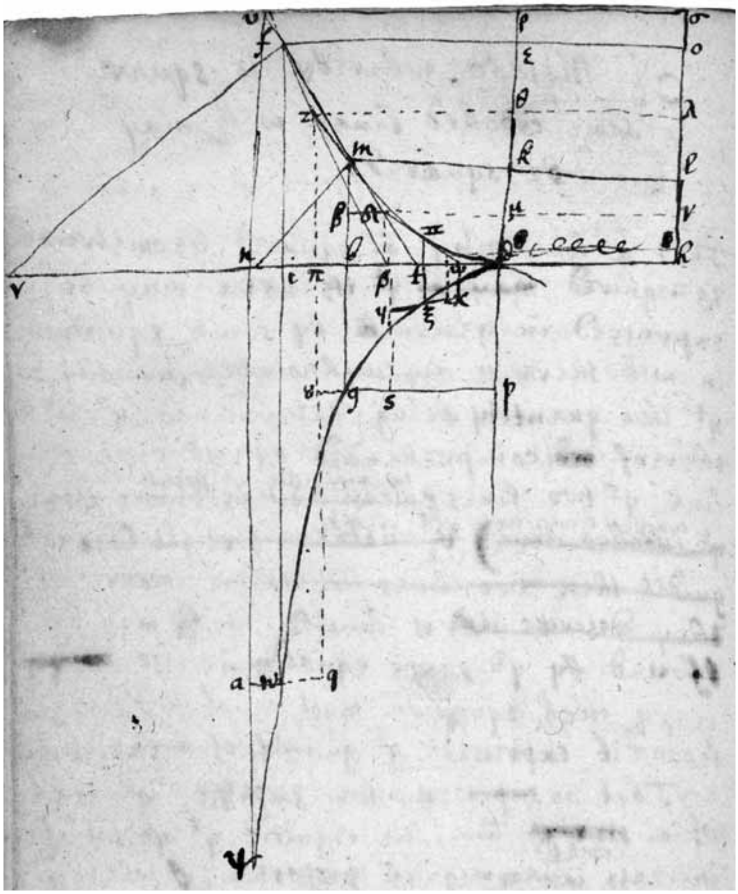


Figure 8.7
Newton on the relation between area-problems and tangent-problems (c. 1665). Source: Add. 4000, f. 120v. Reproduced by kind permission of the Syndics of Cambridge University Library.

where bg is an ordinate of the curve y, βm and $\Omega\beta$ are infinitesimal increments of z and x, and dh is a unit length segment. It immediately follows that the area $bpsg$ ($= \Omega\beta \cdot bg$) and the area $\mu\kappa\lambda\nu$ ($= \beta m \cdot dh$) are equal. It was commonplace in seventeenth-century mathematics to consider the surface subtended by a curve to be equal to the juxtaposition of infinitely many infinitesimal rectangles such as $bpsg$. It follows that the curvilinear area subtended by y, for instance, $d\psi n$, is equal to the rectangular area $dh\sigma\rho$. A knowledge of z then makes it possible to "square" y, since the area under y (the derivative curve) is proportional to the difference between corresponding ordinates of z.[36] In Leibnizian terms, Newton proved that the integral of the differential of z is equal to z, namely, $\int dz = z$. A proof of the fact that $d \int z = z$ can be found in *De Analysi*.

Proof in *De Analysi* (1669) A proof of the inverse relation of area-problems and tangent-problems is given at the end of *De Analysi*. Newton proceeded as follows. He considered a curve $AD\delta$ (figure 8.8), where $AB = x$, $BD = y$, and the area $ABD = z$. He defined $B\beta = o$ and $BK = v$, so that "the rectangle $B\beta HK$ $(= ov)$ is equal to the space $B\beta\delta D$."[37] Further, Newton assumed that $B\beta$ is infinitely small (*infinite parvam*).

Given these definitions, $A\beta = x + o$ and the area $A\delta\beta$ is equal to $z + ov$. At this point Newton wrote, "[F]rom any arbitrarily assumed relationship between x and z I seek y."[38] He noted that the increment of the area ov divided by the increment of the abscissa o is equal to v. But since one can assume "$B\beta$ to be infinitely small, that is, o to be zero, v and y will be equal."[39] Therefore, the rate of increase of the area is equal to the ordinate. The mathematically trained reader will notice that several assumptions that tacitly operate in this reasoning (most notably, the existence of v) only received attention and were systematized in the nineteenth century.

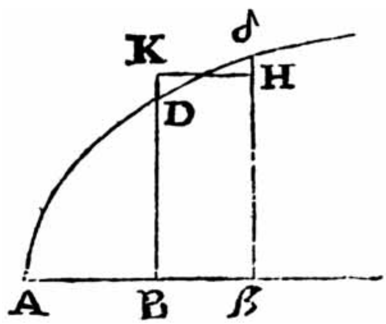


Figure 8.8
Newton on the relations between area-problems and tangent-problems, from *De Analysi* (1669). Source: Newton, *Analysis per Quantitatum* (1711), p. 19. Courtesy of the Biblioteca Universitaria di Bologna.

[36] As Westfall states in *Never at Rest* (1980), p. 127.
[37] MP, 2, p. 243. "rectangulum $B\beta HK$ $(= ov)$ aequale spatio $B\beta\delta D$." MP, 2, p. 242.
[38] MP, 2, p. 243. "ex relatione inter x & z ad arbitrium assumpta quaero y." MP, 2, p. 242.
[39] MP, 2, p. 243. "$B\beta$ esse infinite parvam, sive o esse nihil, erunt v & y aequales." MP, 2, p. 242.

8.3 The Direct Method of Fluxions

8.3.1 Problem 1 in *De Methodis*

I now turn to the two problems into which Newton's method is subdivided, in this section the direct Problem 1, and in subsequent sections the more difficult inverse Problem 2.

Problem 1 is stated as follows:

> Given the relation of the flowing quantities to one another, to determine the relation of the fluxions.[40]

Newton presented the basic algorithm for Problem 1 by providing some examples.[41] First he dealt with polynomial equations. Then he considered equations in which "fractions and surd quantities are present."[42] Last, he considered the case of "quantities which cannot be determined and expressed by any geometrical ratio [*nulla ratione geometrica*], such as the areas and lengths of curves."[43]

8.3.2 Fluxions of Polynomial Equations in *De Methodis*

In *De Methodis* Newton dealt with the equation of a cubic curve:

$$x^3 - ax^2 + axy - y^3 = 0. \tag{8.2}$$

His (inaccurate) prescription is as follows:

> Arrange the equation by which the given relation is expressed according to the dimensions of some fluent quantity, say x, and multiply its terms by any arithmetical progression and then by $\dot{x}/x$. Carry out this operation separately for each one of the fluent quantities and then put the sum of all the products equal to nothing, and you have the desired equation.[44]

This is basically the rule explained in one of the appendices to Descartes' *Géométrie* by Johan Hudde. The application that follows illustrates what Newton meant far more clearly. He obtained:

$$3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + a\dot{y}x - 3\dot{y}y^2 = 0. \tag{8.3}$$

---

[40] MP, 3, p. 75.

[41] MP, 3, pp. 75–9.

[42] MP, 3, p. 77.

[43] MP, 3, p. 79. Newton was aware of the fact that while the direct algorithm applied to geometrical curves generates well-known and easier geometrical curves, the inverse algorithm can lead to mechanical curves. So, for instance, the area subtended to the hyperbola is expressed by logarithms, and the arclength or the sector of a circle by trigonometric magnitudes.

[44] MP, 3, p. 75. In applying the rule, Newton chose the simplest arithmetical progression $3, 2, 1, 0$. The same results obtain, for instance, for $6, 4, 2, 0$. With the latter choice one would obtain $6\dot{x}x^2 - 4a\dot{x}x + 2a\dot{x}y + 2a\dot{y}x - 6\dot{y}y^2 = 0$.

This equation gives the ratio of the fluxions of, $\dot{x}$ to $\dot{y}$, as

$$\frac{3y^2 - ax}{3x^2 - 2ax + ay}. \tag{8.4}$$

Note that in the example the rules for the calculation of the fluxions of the sum $x+y$, the product xy, and the integer positive power x^n are simultaneously stated, respectively, as $\dot{x} + \dot{y}$, $x\dot{y} + y\dot{x}$, and $nx^{n-1}\dot{x}$. Naturally, the fluxion of a constant quantity is equal to zero.[45] It was unclear how to extend this rule to nonpolynomial equations until Newton provided an answer.

8.3.3 Fluxions of Complex Fractions and Surd Quantities in *De Methodis*

In *De Methodis*, Newton wrote,

> Whenever complex fractions or surd quantities are present in the proposed equation, in place of each I put a corresponding letter and, supposing these to designate fluent quantities, I work as before. Then I suppress and exterminate the letters ascribed.[46]

Take $y^2 - a^2 - x\sqrt{a^2 - x^2} = 0$. Newton set $z = x\sqrt{a^2 - x^2}$ and so obtained $y^2 - a^2 - z = 0$ and $a^2x^2 - x^4 - z^2 = 0$. Applying the direct algorithm for polynomial equations, he determined $2\dot{y}y - \dot{z} = 0$ and $2a^2\dot{x}x - 4\dot{x}x^3 - 2\dot{z}z = 0$. He then eliminated $\dot{z}$, restored $z = x\sqrt{a^2 - x^2}$, and thus obtained $2\dot{y}y + (-a^2\dot{x} + 2\dot{x}x^2)/\sqrt{a^2 - x^2} = 0$ as the sought relation between $\dot{y}$ and $\dot{x}$. In this first example proposed by Newton, of course, the radical could easily be eliminated. But Newton's procedure is a very effective method for the calculation of fluxions in more complicated cases. In practice, by substitution of a variable it is possible to eliminate radicals (and quotients, of course) and thus apply Hudde's rule, which is valid for polynomial equations.

Another example will help to illustrate Newton's procedure. Let the relation between the fluents be

$$y = \sqrt{a + bx + cx^2} + \frac{1}{\sqrt{dx + ex^2}}. \tag{8.5}$$

Set

$$y = y_1 + y_2 = \sqrt{z} + \frac{1}{\sqrt{w}}, \tag{8.6}$$

with

$$z = a + bx + cx^2 \tag{8.7}$$

[45] Leibniz, some fifteen years later, was able to state and formulate these rules in a much clearer form (see chapter 17).
[46] MP, 3, p. 77.

and

$$w = dx + ex^2. \tag{8.8}$$

Applying Hudde's rule to these polynomial equations,

$$\dot{z} = b\dot{x} + 2cx\dot{x} \tag{8.9}$$

and

$$\dot{w} = d\dot{x} + 2ex\dot{x}. \tag{8.10}$$

Of course,

$$\dot{y} = \dot{y_1} + \dot{y_2}, \tag{8.11}$$

as Newton stated (§8.3.2). By applying Hudde's rule to $y_1^2 = z$ and $wy_2^2 = 1$,

$$\dot{y} = \dot{y_1} + \dot{y_2} = \frac{\dot{z}}{2\sqrt{z}} - \frac{\dot{w}}{2w^{3/2}}. \tag{8.12}$$

Substitution for z, w, $\dot{z}$, and $\dot{w}$ delivers the sought ratio of the fluxions ($\dot{y}$ to $\dot{x}$):

$$\frac{b + 2cx}{2\sqrt{a + bx + cx^2}} - \frac{d + 2ex}{2\sqrt{(dx + ex^2)^3}}. \tag{8.13}$$

8.3.4 Fluxions of Nongeometrical Equations in *De Methodis*

I have shown how Newton dealt with the calculation of the relation between the fluxions when the relation between the fluents is expressed by an equation involving quotients and radicals. In *De Methodis* he considered a more difficult case:

> To be sure, even if quantities be involved in an equation which cannot be determined and expressed by any geometrical technique, such as the areas and lengths of curves, the relations of the fluxions are still to be investigated the same way.[47]

The relation of the fluxions can in fact immediately be calculated by applying the fundamental theorem.[48]

One of Newton's examples in the *De Methodis* is

$$z^2 + axz - y^4 = 0, \tag{8.14}$$

---

[47] MP, 3, p. 79.

[48] In Leibnizian terminology, it is possible to say that it was customary in Newton's time to think about transcendental curves as obtained by integrating algebraic curves, so that if $y(x)$ is algebraic, the integral $z = \int_a^x y dx$ can lead to a new class of transcendental curves. The prototypic example can be the logarithm obtained by integrating $y = 1/x$.

where z is the area of the segment ABD of a circle whose diameter is a, abscissa $AB = x$, and ordinate $BD = \sqrt{ax - x^2}$ (see figure 7.5). From equation (8.14) one gets

$$2\dot{z}z + a\dot{z}x + a\dot{x}z - 4\dot{y}y^3 = 0. \tag{8.15}$$

From the fundamental theorem, the fluxion of the area ABD is equal to the length of the ordinate BD times the fluxion of the abscissa:

$$\dot{z} = \dot{x}\sqrt{ax - x^2}. \tag{8.16}$$

Thus, for the relation of the fluxions $\dot{x}$ and $\dot{y}$,

$$(2\dot{x}z + a\dot{x}x)\sqrt{ax - x^2} + a\dot{x}z - 4\dot{y}y^3 = 0. \tag{8.17}$$

Note that in (8.17) binomial expansion is necessary in order to determine z. One must expand $\sqrt{ax - x^2}$ in the right-hand term of (8.16) and integrate (in Leibnizian terms) term-wise.

8.3.5 Demonstration of the Direct Method in *De Methodis*

Newton's procedure for Problem 1 is algebraic. In *De Methodis* he presented his method as a series of algorithmic procedures that are explained by particular examples. The style is didactic, heuristic, and algebraic. This is perfectly in line with the seventeenth-century tradition of the analytical school embodied by Oughtred and Wallis. However, in a section of *De Methodis*, Newton included a demonstration of these rules based on a reasoning that was strongly reminiscent of Barrow's determination of tangents *ex calculo* (§8.1.4). Newton wrote,

> The moments of fluent quantities (that is, their indefinitely small parts, by addition of which they increase during each infinitely small period of time) are as their speeds of flow. Wherefore if the moment of any particular one, say x, be expressed by the product of its speed $\dot{x}$ and an infinitely small quantity o (that is, by $\dot{x}o$), then the moment of the others v, y, z, will be expressed by $\dot{v}o$, $\dot{y}o$, $\dot{z}o$ . . . . Consequently, an equation which expresses a relationship of fluent quantities without variance at all times will express that relationship equally between $x+\dot{x}o$ and $y+\dot{y}o$ as between x and y; and so $x+\dot{x}o$ and $y+\dot{y}o$ may be substituted in place of the latter quantities, x and y, in the said equation.[49]

Let us reconsider equation (8.2). From what has been said, it is permissible to substitute $x + \dot{x}o$ in place of x, and $y + \dot{y}o$ in place of y. Next, Newton deleted $x^3 - ax^2 + axy - y^3$ as equal to zero, and after division by o he obtained an equation from which he canceled the terms with o as a factor. These terms "will be equivalent

[49] MP, 3, pp. 79, 81.

to nothing in respect of the others" since "o is supposed to be infinitely small."[50] This procedure leads straight to Hudde's rule.

This demonstration is achieved through two steps. The first step assumes that it is possible to substitute $x+\dot{x}o$ in place of x, and $y+\dot{y}o$ in place of y. Here Newton meant that the relation valid for the fluents x and y, expressed by an equation, continues to be valid for the values $x+\dot{x}o$ and $y+\dot{y}o$ obtained after momentary increases. In geometrical terms, if the point (x, y) is on the curve, then the infinitely close point $(x+\dot{x}o, y+\dot{y}o)$ will also be on the curve. The latter step is a rule of cancellation of higher-order infinitesimals (equivalent to Leibniz's $x+dx=x$). According to this rule, if x is finite and o is an infinitesimal interval of time, then $x+\dot{x}o=x$. Newton set out to justify the use of infinitesimals in an Addendum to *De Methodis* that he drafted in 1671 (see chapter 9).

8.3.6 Determination of Tangents in *De Methodis*

Determination of tangents: The method The algorithm for Problem 1 allows the resolution of several geometrical problems: the determination of maxima and minima (Problem 3), the determination of tangents (Problem 4), the determination of curvature (Problems 5 and 6).

How does the algorithm work for tangents? Newton assumed that ED is a given curve and that an equation relating the abscissa $x=AB$ to the oblique ordinate $y=BD$ is given (figure 8.9). Let the ordinate BD "move through an indefinitely small space to the position $b\partial$ so that it increases by the moment $c\partial$ while AB increases by the moment Bb equal to Dc."[51] The momentary increases $Bb=Dc$ and $c\partial$ are indicated in the figure. Newton stated that the straight line that prolongs the momentary increase $D\partial$ of the arc ED cuts the axis of the abscissae in T and that this straight line (namely, the tangent) will touch the curve in D and ∂. Without further explanation, Newton stated that the triangle $Dc\partial$ is similar to the triangle TBD, where TB is the subtangent.[52] Therefore, he deduced that

$$\frac{TB}{BD}=\frac{Bb}{c\partial}. \tag{8.18}$$

Given this premise, the subtangent will be found by application of the algorithm for Problem 1 to the equation that defines the curve ED. Indeed, the algorithm

[50] MP, 3, p. 81. "respectu caeterorum nihil valebunt," "o supponitur esse infinite parvum." MP, 3, p. 80.
[51] MP, 3, p. 123.
[52] The subtangent is defined as the segment of the x-axis lying between the x-coordinate of the point at which a tangent is drawn to a curve (in figure 8.9, B) and the intercept of the tangent with the x-axis (in figure 8.9, T).

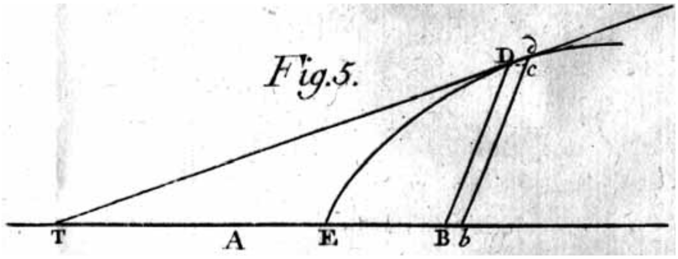


Figure 8.9
Momentary increases for the method of tangents, from *De Methodis* (1671). Source: Newton, *Opuscula Mathematica, Philosophica et Philologica* (1744), 1, Tab. I. Courtesy of the Biblioteca Angelo Mai (Bergamo).

allows the determination of the ratio between the fluxions of x and y. Since the momentary increases are as the fluxions (§8.2.2), it is possible to conclude that the sought subtangent TB is given by[53]

$$TB = y\frac{\dot{x}}{\dot{y}}. \tag{8.19}$$

Note that Newton was deployed well-established practices in handling infinitely or indefinitely small quantities. His use of the indefinitely small triangle $Dc\partial$ is very similar to Barrow's (§8.1.4). This calculation of tangents is like Barrow's in character, as is the notion that magnitudes are generated by motion.

Already while composing *De Methodis*, Newton was aware that some firmer foundation for infinitesimal techniques had to be sought. He found it in a theory of limits that he termed the "method of first and ultimate ratios." Newton developed this method in the 1680s, but its roots were already discernible in the Addendum to *De Methodis*.

Determination of tangents: An example I now consider an example of Newton's method for tangents: the conchoid. Let ED be the conchoid. G is the pole, AT the asymptote. Recall that given a line AT and a line bundle passing through the pole G, the curve is constructed by placing at both sides of AT a distance $LD = L\partial$ on all lines. The two branches of the conchoid are the loci of points D and ∂ (figure 8.10).

Let $GA = b$, $LD = c$, $AB = x$, and $BD = y$. Because the triangles DBL and GDM are similar, $LB/BD = DM/MG$, that is, $\sqrt{c^2 - y^2} : y = x : (b+y)$. Newton wrote,

$$yx = (b + y)\sqrt{c^2 - y^2}. \tag{8.20}$$

[53] I alter Newton's notation that expresses a proportionality: $n.m :: BD.TB$. MP, 3, p. 122.

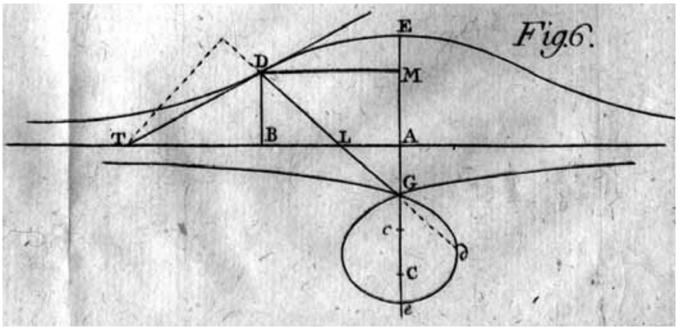


Figure 8.10
Tangent to the conchoid, from *De Methodis* (1671). Source: Newton, *Opuscula Mathematica, Philosophica et Philologica* (1744), 1, Tab. I. Courtesy of the Biblioteca Angelo Mai (Bergamo).

Now he applied his algorithm for "surd" quantities (§8.3.3) and set $z = \sqrt{c^2 - y^2}$. This leads to the following system:

$$yx = bz + yz \tag{8.21}$$
$$z^2 = c^2 - y^2. \tag{8.22}$$

By application of the algorithm for Problem 1, one obtains

$$\dot{x}y + \dot{y}x = b\dot{z} + \dot{y}z + \dot{z}y \tag{8.23}$$
$$\dot{z}z = -\dot{y}y. \tag{8.24}$$

Elimination of $\dot{z}$ leads to

$$\dot{x}y + \dot{y}x = -\frac{b\dot{y}y}{z} - \frac{\dot{y}y^2}{z} + \dot{y}z; \tag{8.25}$$

therefore the ratio between the fluxions can be expressed as

$$\frac{\dot{x}}{\dot{y}} = \frac{z - x - (by + y^2)/z}{y}. \tag{8.26}$$

From equation (8.19) one gets

$$TB = y\frac{\dot{x}}{\dot{y}} = z - x - \frac{y(b+y)}{z}. \tag{8.27}$$

As previously noted (§3.2.2), for Newton's contemporaries a geometrical problem was solved by a geometrical construction, not by an algebraic formulation. Accordingly Newton interpreted this result in geometrical terms:

$$-TB = AL + \frac{BD \times GM}{BL}. \tag{8.28}$$

Newton applied the direct algorithm for Problem 1 to other problems concerning tangency and curvature, as he continued *De Methodis* by considering different coordinate systems (e.g., polar and bipolar coordinates) and developed the theory of curvature in great detail. The determination of the radius of curvature of plane curves was of great importance for Newton in his study of trajectories in the *Principia*, since he made use of the fact that the normal component of the force [F_N] acting on a mass point is proportional to the square of speed [v] and inversely proportional to the radius of curvature [$F_N \propto v^2/\rho$].

8.4 The Inverse Method of Fluxions

8.4.1 Problem 2

Problem 2 is worded as the inverse of Problem 1:

> When an equation involving the fluxions of quantities is exhibited, to determine the relation of the quantities one to another.[54]

Problem 2 was often referred to by Newton as the problem of the quadrature of curves or squaring of curves (§8.2.5). In Leibnizian terms, Newton posed the problem of integration.

Problem 2 is of course much more difficult than Problem 1. Here Newton stopped teaching his method to *discentes* (learners) and addressed the *artifices* (skilled practitioners). The distinction between the parts of his method within reach of the learners and those accessible only to skilled practitioners was quite clear in Newton's mind.[55] As with his treatment of Problem 1, Newton explained how to deal with Problem 2 via examples. His strategy was fragmentary and his style that of the craftsman seeking to make a novice become used to increasingly complex cases. Newton's main techniques for Problem 2 are the following three methods.

8.4.2 Method 1: Squaring of Curves by Series Expansions

The first method was discussed in chapter 7. In Leibnizian terms, it consists in expanding the integrand into a power series. Newton deployed his algorithmic techniques of series expansion by long division, root extraction, and resolution of affected equations.

In a simpler case (Case 1) one has an equation in which "two fluxions together with one only of their fluent quantities are involved." In Leibnizian terms, Newton

[54] MP, 3, p. 83.

[55] MP, 3, p. 85. Newton began *De Methodis* by addressing himself to *discentes*, MP, 3, p. 32. From Problem 2 on, he addressed himself to *artifices*. MP, 3, p. 84.

was here considering ordinary differential equations.[56] From Newton's examples,[57] one takes

$$(\dot{y}/\dot{x})^3 + ax(\dot{y}/\dot{x}) + a^2(\dot{y}/\dot{x}) - x^3 - 2a^3 = 0. \tag{8.29}$$

Applying his technique for the resolution of affected equations (figure 7.11), Newton obtained

$$\frac{\dot{y}}{\dot{x}} = a - \frac{x}{4} + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3} + \cdots, \tag{8.30}$$

which can be squared term-wise, thus obtaining the relation between the fluents:

$$y = ax - \frac{x^2}{8} + \frac{x^3}{192a} + \frac{131x^4}{2048a^2} + \cdots. \tag{8.31}$$

This is one of the basic techniques of series expansion employed in *De Analysi.* It should be noted that the approximation is valid for $x \approx 0$: Newton, that is, obtained a local approximation of the fluent (the integral, in Leibnizian terms).

In slightly more complex cases, an equation is given in which either two fluxions $\dot{x}$ and $\dot{y}$ together with both the fluent quantities x and y (Case 2) occur, or more than two fluxions are present (Case 3). Newton here implemented an algorithm of successive approximations (figure 8.11), where again the aim was to express $\dot{y}/\dot{x}$ as an infinite series.

8.4.3 Method 2: Squaring of Curves by Means of Finite Equations

Two further approaches to Problem 2 were at Newton's disposal:

> Hitherto we have exposed the quadrature of curves defined by less simple equations by the technique of reducing them to equations consisting of infinitely many simple terms [Method 1]. However, curves of this kind may sometimes be squared by means of finite equations also [Method 2], or at least compared with other curves (such as conics) whose area may, after a fashion, be accepted as known [Method 3]. For this reason I have now decided to add the two following catalogues of theorems constructed ... for this use with help of Problems 7 and 8.[58]

[56] MP, 3, p. 91.
[57] MP, 3, pp. 89–91.
[58] MP, 3, p. 237. "Hactenus Curvarum quae per aequationes minus simplices definiuntur Quadraturam mediante reducione in aequationes ex infinite multis terminis simplicibus constantes ostendimus. Cum vero ejusmodi curvae per finitas etiam aequationes nonnunquam quadrari possint vel saltem comparari cum alijs curvis quarum areae quodammodo pro cognitis habeantur, quales sunt sectiones conicae: eapropter sequentes duos Theorematum catalogos in illum usum ope Propositionis 7^{æ} & 8^{æ} ut promisimus constructos, jam visum est adjungere." MP, 3, p. 236.

	$+1.$
$+\frac{y}{a}$	$* + \frac{x}{a} + \frac{xx}{2aa} + \frac{x^3}{2a^3} + \frac{x^4}{2a^4} + \frac{x^5}{2a^5}$; &c.
$+\frac{xy}{aa}$	$* \quad * + \frac{xx}{aa} + \frac{x^3}{2a^3} + \frac{x^4}{2a^4} + \frac{x^5}{2a^5}$; &c.
$+\frac{xxy}{a^3}$	$* \quad * \quad * + \frac{x^3}{a^3} + \frac{x^4}{2a^4} + \frac{x^5}{2a^5}$; &c.
$+\frac{x^3y}{a^4}$	$* \quad * \quad * \quad * + \frac{x^4}{a^4} + \frac{x^5}{2a^5}$; &c.
$+\frac{x^4y}{a^5}$	$* \quad * \quad * \quad * \quad * + \frac{x^5}{a^5}$; &c.
&c.	
Aggregat.	$1 + \frac{x}{a} + \frac{3xx}{2aa} + \frac{2x^3}{a^3} + \frac{5x^4}{2a^4} + \frac{3x^5}{a^5}$; &c.
$y =$	$x + \frac{xx}{2a} + \frac{x^3}{2aa} + \frac{x^4}{2a^3} + \frac{x^5}{2a^4} + \frac{x^6}{2a^5}$; &c.

Figure 8.11

Term-wise solution of Problem 2, Case 2, from *De Methodis* (1671). The fluxional equation is $\dot{y}/\dot{x} = 1 + y/a + xy/a^2 + x^2y/a^3 + x^3y/a^4 + \cdots$. Source: Newton, *Opuscula Mathematica, Philosophica et Philologica* (1744), 1, p. 73. Courtesy of the Biblioteca Angelo Mai (Bergamo).

Newton therefore distinguished between three quadrature techniques:

1. Squaring by reduction to equations consisting of infinitely many simple terms.
2. Squaring by means of finite equations.[59]
3. Squaring by comparison with other curves (such as conics).

The first technique is treated in *De Analysi* (§7.4) and further developed in Problem 2 (Cases 1, 2, and 3) of *De Methodis* (§8.4.2).

The second approach is studied in Problem 7 of *De Methodis*, and its application translated into a first catalogue of curves.

The third approach is studied in Problem 8 of *De Methodis*, and its application translated into a second catalogue of curves.

The second approach consists in applying the algorithm of Problem 1 to "any equation at will defining the relationship of t [the area] to z [the abscissa]" (figure 8.12). One thus obtains an equation relating $\dot{t}$ and $\dot{z}$, and so "two equations will be had, the latter of which will define the curve [whose ordinate is y], the former its area."[60] Following this strategy, Newton constructed a first "catalogue of curves

[59] MP, 3, p. 237. This method is hinted at in *De Analysi* under the rubric "inventio curvarum quae possunt quadrari." MP, 2, p. 244.

[60] MP, 3, p. 197.

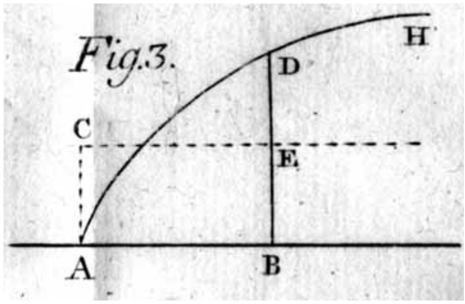


Figure 8.12
Relations between abscissa z $(= AB)$, ordinate y $(= BD)$, and area t $(= ADB)$ in Problem 7, from *De Methodis* (1671). The flow of the ordinate BD generates the surface ADB. Newton proved that $\dot{t}/\dot{z} = y/1$. This becomes $\dot{t} = y$ for $\dot{z} = 1$. Source: Newton, *Opuscula Mathematica, Philosophica et Philologica* (1744), 1, Tab. I. Courtesy of the Biblioteca Angelo Mai (Bergamo).

Curvarum Formæ		Curvarum Areæ
I	$dz^{\eta-1} = y$	$\frac{d}{\eta}z^{\eta} = t$
II	$\frac{dz^{\eta-1}}{e^2 + 2efz^{\eta} + f^2z^{2\eta}} = y$	$\frac{dz^{\eta}}{\eta e^2 + \eta efz^{\eta}} = t$, vel $\frac{-d}{\eta ef + \eta f^2 z^{\eta}} = t$

Figure 8.13
Newton's first catalogue of curves (beginning). d, e, f, g, h are positive constants, η is a "positive or negative, integral or fractional number." Variables z, y, and t denote the curve's abscissa, ordinate, and area, respectively. The first column tabulates the curves' equations, the second column their corresponding areas. It is easy to show that the fluxion of the area t is equal to the ordinate y (assuming that the abscissa flows with constant speed $\dot{z} = 1$). This catalogue was reproduced in *De Quadratura* (1704). Note that in the first species $\eta \neq 0$, since in this case the curve is not quadrable in finite terms. Further, the second species must be set equal to (i) $dz^{\eta-1}/(e + fz^{\eta})^2$ or (multiplying numerator and denominator of (i) by $z^{-2\eta}$) to (ii) $dz^{-\eta-1}/(ez^{-\eta} + f)^2$; thus in the second column we find two different values of the area t. Source: Newton, *Analysis per Quantitatum* (1711), p. 62. Courtesy of the Biblioteca Universitaria di Bologna.

which can be squared by means of finite equations" (figure 8.13). In modern terms, one might say that Newton was aware of the fact that antiderivatives are related to definite integrals through the fundamental theorem of calculus and provide a convenient means for tabulating the integrals of many functions.

8.4.4 Method 3: Squaring of Curves by Comparison with Conic Sections

Curves "which can be squared by means of finite equations" are an exception: infinite series remain an essential tool for calculating many curvilinear areas. Most often these series are difficult to interpret geometrically and provide only a local and algorithmic approximation. Recall that for Newton the result of an analytical process is best interpreted geometrically. This is why, in Method 3, Newton considered transformations of variables to reduce the calculation of a curvilinear surface to the calculation of the area of a conic surface. Conic areas can be evaluated by binomial expansion and term-wise quadrature, as Newton explained in *De Analysi*. Therefore, series are still necessary. However, the areas of the conic sections can be considered to be accepted as known, not only because their areas are given by well-known logarithmic and trigonometric tables but also because the conics are geometrically constructible following methods already established in Antiquity.

In Problem 8, Newton took two curves FDH and GEI, in which variables x, v, s and z, y, t denote the abscissa, ordinate, and area of the two curves (figure 8.14). Suppose one knows how to square the curve FDH. The problem here will be to square GEI. Newton introduced two equations, the first relating the abscissae x and z, and the second relating areas s and t. Newton proceeded by examples where the curve FDH is a conic section (which can be squared following the procedures of *De Analysi*). A few simple examples follow.

Let the curve FDH be a circle whose equation is $v^2 = ax - x^2$. Assume that areas s and t are related by

$$cx + s = t, \tag{8.32}$$

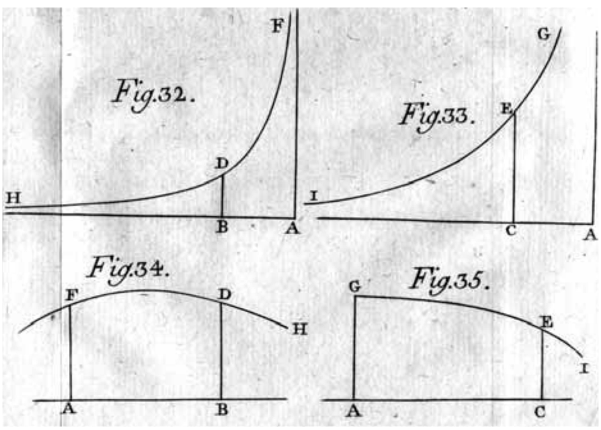


Figure 8.14
Relations between variables $x = AB$, $v = BD$, $s = AFDB$, and $z = AC$, $y = CE$, $t = AGEC$ in Problem 8, from *De Methodis* (1671). Newton proved that $s = t$ when $v/y = \dot{z}/\dot{x}$. As stated in Proposition 9, Theorem 7, of *De Quadratura*, "The Areas of those curves are equal among themselves, whose Ordinates are reciprocally as the Fluxions of their Abscisses. For the Rectangles contain'd under the Ordinates, and the Fluxions of the Abscisses will be equal and the Fluxions of the Areas are as these Rectangles." In symbols $\dot{s} = v\dot{x}$ and $\dot{t} = y\dot{z}$. Source: Newton, *Opuscula Mathematica, Philosophica et Philologica* (1744), 1, Tab. V. Courtesy of the Biblioteca Angelo Mai (Bergamo).

and that the ordinates are related by

$$ax = z^2. \tag{8.33}$$

By means of the algorithm of Problem 1, under the assumption that $\dot{x} = 1$, one gets

$$c + \dot{s} = \dot{t} \tag{8.34}$$

and

$$a = 2\dot{z}z. \tag{8.35}$$

Therefore

$$y = \frac{\dot{t}}{\dot{z}} = \frac{2}{a}z(c + \dot{s}), \tag{8.36}$$

and "this when $\sqrt{ax - x^2}$ is substituted in place of $\dot{s}$ and z^2/a in place of x" becomes

$$y = \frac{2cz}{a} + \frac{2z^2}{a^2}\sqrt{a^2 - z^2}. \tag{8.37}$$

One begins then with a curve (in this case, a circle) whose area is assumed as known and by suitable transformations of variables obtains equation (8.37) of a curve whose area t is related to the area s of the circle by equation (8.32).[61]

Thanks to this technique Newton could develop a "second catalogue of curves related to conic sections" (figure 8.15). The first row states that the area t under

$$y = \frac{dz^{\eta-1}}{e + fz^{\eta}} \tag{8.38}$$

is equal to

$$t = \frac{1}{\eta}s, \tag{8.39}$$

where s is the area under

$$v = d/(e + fx).$$

The second column prescribes a substitution of variables:

$$z^{\eta} = x; \tag{8.40}$$

[61] MP, 3, p. 199. It might be helpful to translate this procedure into familiar Leibnizian notation. Calculate $t = \int ydz = \int(2cz/a + 2z^2/a^2\sqrt{a^2 - z^2})dz$. Substitution of variables $ax = z^2$ leads to $t = \int cdx + \int \sqrt{ax - x^2}dx = cx + s + C$.

Formæ Curvarum.		Sectionis Conicæ.		Arearum Valor.	Fig.
		Abſciſſæ.	*Ordinatæ.*		
I	1 $\dfrac{dz^{\eta-1}}{e+fz^{\eta}}=y$	$z^{\eta}=x$	$\dfrac{d}{e+fx}=u$	$\dfrac{1}{\eta}s=t=\dfrac{\alpha GDB}{\eta}$	N°. 1.
	2 $\dfrac{dz^{2\eta-1}}{e+fz^{\eta}}=y$	$z^{\eta}=x$	$\dfrac{d}{e+fx}=u$	$\dfrac{d}{\eta f}z^{\eta}-\dfrac{e}{\eta f}s=t$	
	3 $\dfrac{dz^{3\eta-1}}{e+fz^{\eta}}=y$	$z^{\eta}=x$	$\dfrac{d}{e+fx}=u$	$\dfrac{d}{2\eta f}z^{2\eta}-\dfrac{de}{\eta ff}z^{\eta}+\dfrac{ee}{\eta ff}s=t$	

Figure 8.15

Newton's second catalogue of curves (beginning). The third column lists the equations of conic sections, whose area s is assumed as given (it must be calculated by power series expansion). The second column gives a transformation of the ordinates x and z, the fourth column that of areas s and t. The first column lists the equation of the curves whose areas can be related to the areas of conic sections. This catalogue was reproduced in *De Quadratura* (1704). Source: Newton, *Analysis per Quantitatum* (1711), p. 101 facing. Courtesy of the Biblioteca Universitaria di Bologna.

assuming $\dot{x} = 1$, one gets

$$\dot{z} = \frac{1}{\eta z^{\eta-1}}. \tag{8.41}$$

From the third column

$$\dot{s} = v = \frac{d}{e + fx} = \frac{d}{e + fz^{\eta}}. \tag{8.42}$$

Therefore, from the first column[62]

$$\dot{t} = y\dot{z} = \frac{dz^{\eta-1}}{e+fz^{\eta}} \frac{1}{\eta z^{\eta-1}} = \frac{1}{\eta}\dot{s}. \tag{8.43}$$

8.4.5 The Analytical Quadrature of the Cissoid

Newton applied the second catalogue of curves to several examples. Example 3 concerns the quadrature of the cissoid. The problem is how to square the cissoid AeE, that is, how to determine the area of the surface $ACEeA$, where ADQ is a circle (figure 8.16).

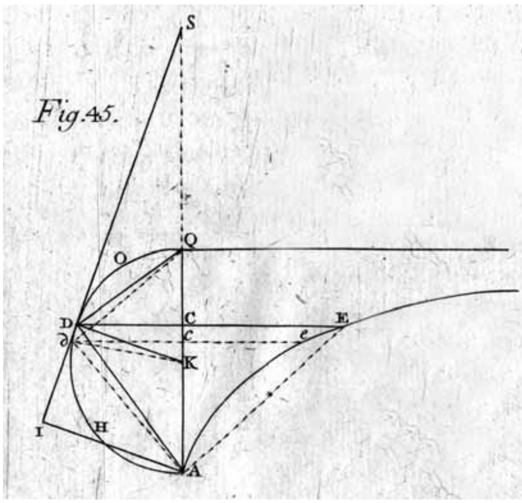


Figure 8.16
Cissoid AeE, from *De Methodis* (1671). Source: Newton, *Opuscula Mathematica, Philosophica et Philologica* (1744), 1, Tab. VII. Courtesy of the Biblioteca Angelo Mai (Bergamo).

[62] I translate Newton's calculation in Leibnizian notation as follows, but note that d is a constant. Eliminate d by setting $d = 1$. Then $t = \int y dz = \int (z^{\eta-1}/(e+fz^{\eta}))dz$. Substitution of $z^{\eta} = x$ leads to $t = (1/\eta) \int (1/(e+fx))dx = (1/\eta) \int v dx = s/\eta + C$.

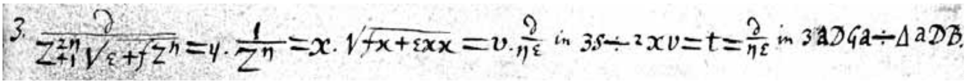


Figure 8.17
Third species of the seventh order of the second catalogue of curves in the manuscript of *De Methodis* (1671). Note that ∂ is a constant, not a partial derivative. Also, Newton did not use the modern symbol for the absolute value $|3s - 2xv|$ but rather one that he found in Barrow's works. Newton wrote $\div$ for "the Difference of two Quantities, when it is uncertain whether the latter should be subtracted from the former, or the former from the latter" (Newton, *Two Treatises* (1745), p. 25). Thus, Newton wrote $3s \div 2xv$. Source: Add. 3960.14, f. 81. Reproduced by kind permission of the Syndics of Cambridge University Library.

Set the abscissa $AC = z$, the ordinate $CE = y$, the circle's diameter $AQ = a$. Because of the defining property of the cissoid, CD, AC, and CE are in continued proportion.[63] Thus, the equation of the cissoid is

$$y = \frac{z^2}{\sqrt{az - z^2}} = \frac{z}{\sqrt{az^{-1} - 1}}. \tag{8.44}$$

In order to square the cissoid, reference must be made to the third species of the seventh order of the second catalogue of curves (figure 8.17).

On setting $\partial = 1$, $\epsilon = -1$, and $f = a$, the curve listed in the first column is

$$y = \frac{z^{-2\eta - 1}}{\sqrt{az^{\eta} - 1}}, \tag{8.45}$$

which is the equation of the cissoid for $\eta = -1$.
The transformation of the abscissae (second column) is $z = z^{-\eta} = x$ (therefore $x = AC$).
The conic ordinate (third column) is $v = \sqrt{ax - x^2}$ (therefore $v = CD$), and s is the area of the segment $ACDH$ of the circle.

From the fourth column one gets that the area t under the cissoid is

$$t = 3s - 2xv, \tag{8.46}$$

consequently, the area $ACEeA$ of the cissoid is $3(ACDH) - 4\triangle ADC$.[64]

Newton added some equivalent formulations: "Or what is the same, $3\times$ segment $ADHA$ = area $ADEA$, that is, $4\times$ segment $ADHA$ = area $AHDEeA$."[65]

---

[63] Namely, $CD/AC = AC/CE$, where $AC = z$, $CE = y$, and $CD = \sqrt{az - z^2}$.

[64] Indeed, since $z = x$, verify by differentiation that
$dt/dz = dt/dx\, dx/dz = d/dx \left(3 \int \sqrt{ax - x^2} dx - 2x\sqrt{ax - x^2} + C\right) = 3\sqrt{ax - x^2} - 2\sqrt{ax - x^2} - 2x(a - 2x)/(2\sqrt{ax - x^2}) = x^2/\sqrt{ax - x^2} = y$.

[65] MP, 3, p. 271.

To conclude, the calculation of the area t of the cissoid is reduced to the calculation of the area s of the circle, which can be evaluated through the power series expansion of $\dot{s}/\dot{x} = v = \sqrt{ax - x^2}$ and term-wise quadrature.

8.5 The Inverse Method in *De Quadratura*

8.5.1 Toward *De Quadratura*

Until the publication of the *Principia*, Newton circulated his mathematical ideas via manuscript exchange and correspondence. This publication practice exposed him to the risk of not having his discoveries recognized (see part VI). In 1685, John Craig published a short treatise on the quadrature of curvilinear figures in which Newton's contributions were just mentioned in passing.[66] More dangerously, David Gregory was claiming for himself a theorem on quadratures that Newton had privately communicated to Leibniz in the *epistola posterior*, dated October 24, 1676, and to Craig, who had visited Newton in his rooms at Trinity in 1685. Through Craig the theorem had passed into Gregory's hands. In 1688, Gregory's associate Archibald Pitcairn had published the theorem attributing it to Gregory.[67] In 1691, after having being elected Savilian Professor of Astronomy at Oxford, Gregory wrote a letter to Newton in which, rather obliquely, he tried to secure the authorship of this important result.[68] Newton reacted by writing a short account of his discoveries on quadratures. He soon changed his mind and set out to write a full-fledged treatise, whose composition probably occupied him in the winter of 1691–1692. This was to become *Tractatus de Quadratura Qurvarum*, eventually printed in 1704 as an appendix to the *Opticks*.

It is worth considering Newton's method of quadrature, as communicated to Leibniz in 1676 and to Craig in 1685. In the *epistola posterior*, he wrote,

> For any curve let $dz^\theta \times (e + fz^\eta)^\lambda$ be the ordinate, standing normal at the end of z of the abscissa or the base, where the letters d, e, f denote any given quantities [N.B d is a constant!], and θ, η, λ are the indices of the powers of the quantities to which they are attached.
>
> Put
>
> $$(\theta + 1)/\eta = r, \ \lambda + r = s, \ (d/(\eta f)) \times (e + fz^\eta)^{\lambda+1} = Q, \ r\eta - \eta = \pi,$$
>
> then the area of the curve will be
>
> $$Q \times \left\{ \frac{z^\pi}{s} - \frac{r-1}{s-1} \times \frac{eA}{fz^\eta} + \frac{r-2}{s-2} \times \frac{eB}{fz^\eta} - \frac{r-3}{s-3} \times \frac{eC}{fz^\eta} + \frac{r-4}{s-4} \times \frac{eD}{fz^\eta}, \, etc. \right\}$$

[66] Craig, *Methodus Figurarum* (1685).

[67] Pitcairne, *Solutio Problematis* (1688). "Gregory's" method of quadrature was also printed in Wallis, *Opera*, 2, pp. 337–80. On this episode, see the commentary by Whiteside in MP, 7, pp. 3–13. On the circumstances surrounding Newton's exchange of letters with Leibniz, see chapter 15.

[68] Gregory to Newton (November 7, 1691). *Correspondence*, 3, pp. 172–6.

> the letters A, B, C, D, etc., denoting the terms immediately preceding; that is A the term z^{π}/s, B the term $-(r-1)/(s-1) \times (eA)/(fz^{\eta})$, etc. This series, when r is a fraction or a negative number, is continued to infinity; but when r is positive and integral it is continued only to as many terms as there are units in r itself; and so it exhibits the geometrical squaring of the curve.[69]

This method of quadrature was proposed to Leibniz as the first of a series of theorems devised in order to simplify the "speculations concerning the squaring curves"; it is thus known in the literature as the prime theorem on quadratures. The prime theorem is a generalization of results contained in the first catalogue of curves of *De Methodis* (§8.4.3).[70]

More generally, Newton was interested in squaring curves of the form $y = z^{\theta} \times (e + fz^{\eta})^{\lambda}$, $y = z^{\theta} \times (e + fz^{\eta} + gz^{2\eta})^{\lambda}$, $y = z^{\theta} \times (e + fz^{\eta} + gz^{2\eta} + hz^{3\eta} \dots)^{\lambda}$, or even $y = z^{\theta} R^{\lambda} S^{\mu} T^{\nu}$, where R, S, T denote expressions of the form $\sum_{i=0}^{\infty} a_i z^{in}$ (to use modern notation). These theorems were systematized in the *De Quadratura* (§8.5.2).

There is little doubt that Newton was keenly aware of the significance of quadrature problems. In the October 1666 tract on fluxions, he had already stated the importance of the inverse problem of fluxions:

> If two Bodys A & B, by their velocitys p & q describe ye lines x & y. & an Equation bee given expressing ye relation twixt one of ye lines x, & ye ratio q/p of their motions p & q; To find the other line y. Could this ever bee done all problems whatever might bee resolved.[71]

Further, in *De Methodis*, he had underlined the importance of quadrature problems:

> Observing that the majority of geometers, with an almost complete neglect of the ancients' synthetic method, now for the most part apply themselves to the cultivation of analysis and with its aid have overcome so many formidable difficulties that they seem to have exhausted virtually everything *apart from the squaring of curves and certain topics of like nature not yet fully elucidated.*[72]

When Newton's polemic with Leibniz broke out, the exchange of accusations between the two was obfuscated by a different perception of what was of primary significance in the discovery of the new method. While Leibniz focused on the enunciation of the rules concerning the direct method of differentiation and—not without reason—claimed that he was the inventor of a simple and concise algorithm

---

69 *Correspondence*, 2, p. 134. Translation by Turnbull. Note that here geometrical is opposed to mechanical: the former means "exactly determined in finite terms," the latter "by approximation via infinite series."
70 See the discussion in MP, 3, p. 237 (n. 540).
71 MP, 1, p. 403.
72 MP, 3, p. 33. Translation by Whiteside. Italics supplied.

for differentiation, Newton insisted on his superior command of series in quadrature techniques (integration, in Leibnizian terms). Newton—never very receptive toward the importance of advances in algorithmic techniques—saw Leibniz's rules for differentiation as mere trivialities. The true, difficult problem, Newton reiterated, was the inverse problem of quadrature: it is on this battleground that— again, not without reason—Newton claimed supremacy over Leibniz. Part 6 expands on these themes.

The problem of squaring ample classes of curves had been beautifully solved by Newton in his *anni mirabiles* by the use of infinite series (Method 1), the fundamental theorem (Method 2), and substitutions of variables (Method 3).

These techniques, didactically presented in *De Methodis*, constitute a method of solution, an heuristic patchwork of algorithmic instructions. In the 1670s, Newton began a research program on quadratures aimed at transforming his early method into a *tractatus*, his early rules into theorems. As he explained to Leibniz in the *epistola posterior* of 1676,

> I have tried also to render the speculations concerning [the method of] squaring curves simpler, and have attained certain general theorems.[73]

This program culminated into *Tractatus de Quadratura Curvarum*, which Newton wrote in 1691–1692.[74] It is *De Quadratura* that Newton chose to print in 1704, not *De Methodis*, which appeared posthumously, in an English translation, only in 1736.

8.5.2 Theorems in *De Quadratura*

Preliminaries *De Quadratura* is a notable work not just for the theorems on the quadrature of curves. In the introductory pages of the work Newton presented a theory of limits that provides a foundation for the method of fluxions, a theory that makes it possible to avoid—so the author claimed—the use of infinitesimals. (The "method of first and ultimate ratios" is discussed in §9.5).

It is in these preliminary pages that Newton also introduced the dotted notation for fluxions ($\dot{x}, \dot{y}$) and slashed notation for fluents ($\acute{x}$, $\acute{y}$). I have adopted this notation here.

---

[73] "Hoc fundamento conatus sum etiam reddere speculationes de Quadratura curvarum simpliciores, pervenique ad Theoremata quaedam generalia." *Correspondence*, 2, p. 115. Later in 1691, Newton transcribed these lines by changing speculationes to methodum. MP, 7, p. 24.

[74] The version composed in the early 1690s was revised for publication in 1703 and appeared under the title of *Tractatus de Quadratura Curvarum* in Newton, *Opticks* (1704), pp. 165–211. The numerous early versions and the revisions (mostly in MSS Add. 3960.7–13, 3962.1–3, and 3965.6 (Cambridge University Library)) are edited in MP, 7, pp. 24–182 and MP, 8, pp. 92–167.

After these important introductory pages devoted to foundations and notation, two themes indicative of Newton's high expectations with regard to this treatise, one finds the treatment of two problems.

The first is Problem 1, on the direct method of fluxions ("having given an equation involving any number of fluents to find their fluxions"). It is essentially a reformulation of Problem 1 of *De Methodis* (§8.3.1). Here, however, terms multiplied by o are discarded not because they are "indefinitely little" but because they are "evanescent." Newton took the limit assuming that "the quantity o is lessened indefinitely" and therefore cancels terms multiplied by it (§9.5).[75]

Problem 2 ("to find curves that are quadrable") is equivalent to Method 2 of *De Methodis* (§8.4.3). One finds here a statement of the fundamental theorem of the calculus. Newton's early proof, based on infinitesimals, of this fundamental relation between a flowing surface and the ordinate that generates it was discussed in section 8.2.6. For the reader's convenience, I present it again by quoting from *De Quadratura*:

> Problem 2: To find the Curves that are Quadrable
> Let ABC be the Figure [8.18] whose Area [t] is to be found; BC [y] an Ordinate apply'd at Right Angles, and AB [z] the Abscissa. Produce CB to E that BE may be = 1, and compleat the Parallelogram $ABED$; and the Fluxions of the Areas

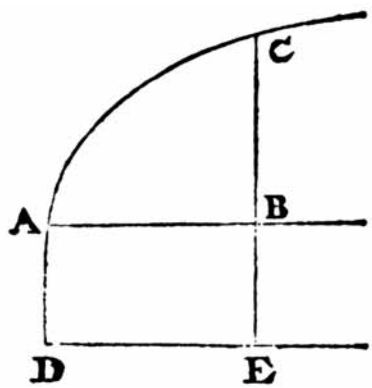


Figure 8.18
Relations between abscissa ($z = AB$), ordinate ($y = BC$), and curvilinear area ($t = ABC$) in Problem 2, from *De Quadratura* (1704). The flow of the ordinate BC generates the surface ABC. Newton proved that $\dot{t}/\dot{z} = y/1$. This becomes $\dot{t} = y$ for $\dot{z} = 1$. Source: Newton, *Analysis per Quantitatum* (1711), p. 48. Courtesy of the Biblioteca Universitaria di Bologna.

[75] Newton, *Mathematical Works* (1964), 1, p. 144. See Newton, *Analysis per Quantitatum* (1711), p. 44.

> ABC, $ABED$ will be as BC to BE: Therefore take any Equation by which the Relation of the Areas may be determined, and thence will be given the Relation of the Ordinates BC and BE.[76]

As in *De Methodis*, Newton's strategy consisted in applying the direct method to equations involving z and t in order to determine the relation between z and y, that is, in order to determine the curves that are "exactly quadrable." In Leibnizian terms, the foundation of this quadrature technique is the inverse relation between differentiation and integration.

A simple example Consider:

$$t = z^{\theta}(1+z)^{\lambda}, \tag{8.47}$$

where θ and λ are integer or fractional.[77] Assuming that the ordinate EBC flows uniformly, that is, $\dot{z} = 1$, one has

$$\frac{\dot{t}}{\dot{z}} = \frac{y}{1} = [\theta + (\theta + \lambda)z]z^{\theta-1}(1+z)^{\lambda-1}. \tag{8.48}$$

Now suppose that one must find the area t of the surface under the curve of equation

$$y = (3 + \frac{7}{2}z)z^2(1+z)^{-1/2}. \tag{8.49}$$

Equation (8.49) can be reduced to the form of equation (8.48) by setting $\theta - 1 = 2$ and $\lambda - 1 = -1/2$, that is, $\theta = 3$ and $\lambda = 1/2$. A lucky coincidence results:

$$\theta + (\theta + \lambda)z = 3 + \frac{7}{2}z. \tag{8.50}$$

Thus one can state that

$$t = z^3(1+z)^{1/2}. \tag{8.51}$$

This curve is indeed "exactly, or geometrically, quadrable."

---

[76] Note that Newton considered the parallelogram of side $AD = 1$ in order to state a proportion in which ratios are established between geometrical magnitudes of equal dimensionality. Newton, *Mathematical Works* (1964), 1, p. 144. See Newton, *Analysis per Quantitatum* (1711), p. 48.

[77] We take it from Dupont, Appunti di Storia di Analisi Infinitesimale (1981–82), 2, p. 539.

Theorem 1 More generally, in Theorem 1, Newton considers curves whose area t is[78]

$$t = z^{\theta} R^{\lambda}, \tag{8.52}$$

where $R = e + fz^{\eta} + gz^{2\eta} + hz^{3\eta} + \cdots$. Therefore,

$$\dot{t} = \theta \dot{z} z^{\theta-1} R^{\lambda} + \lambda z^{\theta} \dot{R} R^{\lambda-1} = z^{\theta-1} R^{\lambda-1}(\theta \dot{z} R + \lambda z \dot{R}). \tag{8.53}$$

But $\dot{R} = \eta f \dot{z} z^{\eta-1} + 2\eta g \dot{z} z^{2\eta-1} + 3\eta h \dot{z} z^{3\eta-1} + \cdots$. Therefore, the curve whose area is equal to equation (8.52) has ordinate y equal to

$$y = \frac{\dot{t}}{\dot{z}} = z^{\theta-1} R^{\lambda-1}[\theta e + f(\theta+\lambda\eta) z^{\eta} + g(\theta + 2\lambda\eta) z^{2\eta} + h(\theta + 3\lambda\eta) z^{3\eta} + \cdots]. \tag{8.54}$$

Theorem 3 An important result is offered in Theorem 3, a generalization of the prime theorem (§8.5.1).[79] Let

$$R = e + fz^{\eta} + gz^{2\eta} + hz^{3\eta} + \cdots . \tag{8.55}$$

Further, set $r = \theta/\eta$, $s = r + \lambda$, $t = s + \lambda$, $v = t + \lambda$, $\ldots$. Then the area t under the curve

$$y = z^{\theta-1} R^{\lambda-1}(a + bz^{\eta} + cz^{2\eta} + dz^{3\eta} + \cdots) \tag{8.56}$$

is

$$\begin{aligned} t = z^{\theta} R^{\lambda}\Bigg(\frac{a/\eta}{re} + \frac{b/\eta - sfA}{(r+1)e} z^{\eta} &\quad + \\ \frac{c/\eta - (s+1)fB - tgA}{(r+2)e} z^{2\eta} &\quad + \\ \frac{d/\eta - (s+2)fC - (t+1)gB - vhA}{(r+3)e} z^{3\eta} &\quad + \quad \cdots \Bigg), \end{aligned}$$

where each A, B, C $\ldots$ is the coefficient of the preceding power of z, namely, $A = (a/\eta)/(re)$, $B = (b/\eta - sfA)/((r+1)e)$, etc.[80]

The procedure followed in order to prove Theorem 3 is the method of undetermined coefficients (figure 8.19). Newton sought the area of the curve with ordinate (8.56) in the form

$$t = z^{\theta} R^{\lambda}(A + Bz^{\eta} + Cz^{2\eta} + Dz^{3\eta} + \cdots). \tag{8.57}$$

---

[78] Newton, *Analysis per Quantitatum* (1711), p. 48.
[79] Newton, *Analysis per Quantitatum* (1711), pp. 49–50.
[80] Newton, *Analysis per Quantitatum* (1711), p. 49 = *Mathematical Works* (1964), 1, p. 145.

Demonstratio.

Sunto juxta Propositionem tertiam.

Curvarum Ordinatæ | Areæ

1 . $\theta e A + \overline{\theta+\lambda\eta} \times fAz^{\eta} + \overline{\theta+2\lambda\eta} \times gAz^{2\eta} + \overline{\theta+3\lambda\eta} \times hAz^{3\eta}$, &c.

2 . . . $+ \overline{\theta+\eta} \times eBz^{\eta} + \overline{\theta+\eta+\lambda\eta} \times fBz^{2\eta} + \overline{\theta+\eta+2\lambda\eta} \times gBz^{3\eta}$ &c.

3 . . . . $+ \overline{\theta+2\eta} \times eCz^{2\eta} + \overline{\theta+2\eta+\lambda\eta} \times fCz^{3\eta}$ &c.

4 . . . . $+ \overline{\theta+3\eta} \times eDz^{3\eta}$ &c.

$\times z^{\theta-1}R^{\lambda-1}$.

$Az^{\theta}R^{\lambda}$, $Bz^{\theta+\eta}R^{\lambda}$, $Cz^{\theta+2\eta}R^{\lambda}$, $Dz^{\theta+3\eta}R^{\lambda}$

Figure 8.19

Demonstration of Theorem 3 of *De Quadratura*. Newton tabulated the partial areas (on the right) and calculated their ordinates as their first fluxions (on the left). The series on the left must be equal to series (8.56). By equating the coefficients one gets $\theta e A = a$. The next step consists in substituting $A = (a/\eta)/(re)$, equating the coefficients, and determining B. Source: Newton, *Analysis per Quantitatum* (1711), p. 50. Courtesy of the Biblioteca Universitaria di Bologna.

Newton next considered the partial areas $Az^{\theta}R^{\lambda}$, $Bz^{\theta+\eta}R^{\lambda}$, $Cz^{\theta+2\eta}R^{\lambda}$, $Dz^{\theta+3\eta}R^{\lambda}$, etc. From the partial areas (by means of the direct method of fluxions) he calculated their respective ordinates, whose sum must be equal to the given expression (8.56) $z^{\theta-1}R^{\lambda-1}(a + bz^{\eta} + cz^{2\eta} + dz^{3\eta} + \cdots)$. In other words, Newton obtained two power series that must be equal and equated the coefficients of the equal powers in η (see figure 8.19). He thus obtained a system of equations in e, f, g, $\ldots$ (the coefficients of R), a, b, c, $\ldots$, and A, B, C, $\ldots$, which can be solved in order to determine A, B, C, $\ldots$. Note that it is the inverse relation between differentiation and integration demonstrated in Problem 2 that justifies this procedure.[81]

A more advanced example In most cases the application of Theorem 3 leads to a calculation of the area as an infinite series. In a few instances, however, the series terminates. Newton gave the following application of Theorem 3.[82] Let

$$y = \frac{3k - lz^2}{z^2\sqrt{kz - lz^3 + mz^4}}; \tag{8.58}$$

[81] The reader interested in the details of the calculation necessary to prove Theorem 3 can consult John Stewart's commentary in Newton, *Two Treatises* (1745), pp. 91–7.

[82] Newton, *Analysis per Quantitatum* (1711), pp. 50–1.

this may be written as

$$y = (3k - lz^2)z^{-5/2}(k - lz^2 + mz^3)^{-1/2}. \tag{8.59}$$

In this case one reduce to (8.56) by setting, for the coefficients of S, $a = 3k$, $b = 0$, $c = -l$, $d = e = f = \cdots = 0$; for the coefficients of R, $e = k$, $f = 0$, $g = -l$, $h = m$, $i = l = m = \cdots = 0$; and finally $\theta = -3/2$, $\lambda = 1/2$, $\eta = 1$. The area will be given in finite terms by

$$t = -2\sqrt{\frac{k - lz^2 + mz^3}{z^3}}. \tag{8.60}$$

As was often the case, Newton assumed that the initial conditions were such that the constant of integration is zero.[83] He also noted that the area is negative because it is "adjacent to the absciss produced beyond the ordinate."[84] Yet, finite quadratures are by no means the rule. In general, the quadrature will be given by an infinite series.

8.5.3 The Style of *De Quadratura*

In concluding this section, I would like to emphasize how general the results on quadratures (namely, integration) reached by Newton in *De Quadratura* actually are. Newton was clearly aiming at expressing results on quadratures in general symbolical terms. Particularly notable is the use of symbols like R for infinite power series. Newton did not illustrate his rules by means of examples, as in *De Methodis*, but rather provided general quadrature theorems concerning ample classes of fluents. Newton's methods allow the integration of all rational functions. It seems to me that in writing *De Quadratura*, Newton was deliberately aiming to achieve a level of generality and deductive order that went beyond the heuristic level of his previous writings on the subject. While *De Quadratura* was published as an appendix to the *Opticks*, the more heuristic *De Methodis* was left in manuscript form during Newton's lifetime. Newton preferred to present to the public at large his more general and abstract treatise on quadratures, rather than his rich but unsystematic method of fluxions.

This said, it should be added that the theorems of *De Quadratura* are statements achieved via Wallisian induction. Newton made this point clear:

> [A]t the start of my mathematical studies I first derived particular quadratures and then by induction arrived at general cases.[85]

---

[83] Newton considered also the case $y = z^{-2}(-l + 3kz^{-2})(m - lz^{-1} + kz^{-3})^{-1/2}$.
[84] Newton, *Two Treatises* (1745), p. 11.
[85] MP, 7, p. 67.

Newton's project to transform the *method* of quadratures into a *theory* was stillborn. This should not be seen as a failure: the integral calculus was, and still is to a certain extent, a matter of art rather than science, a matter of guesswork rather than of algorithmic deduction. One might wonder whether Newton, always careful to meet the high standards of certainty of the ancient synthesis, would have printed his treatise on quadratures had he not been involved in priorities disputes with Gregory and especially with Leibniz (see part VI).

8.6 *Methodus Differentialis*

Since I am dealing with methods for the squaring of curves, a brief mention should be made of the method for the approximation of areas that Newton developed in the mid-1690s in a short treatise entitled "Of Quadrature by Ordinates" in the context of his studies on interpolation.[86] The idea behind this method is that by calculating the $n+1$ values y_i acquired by a fluent $[y = f(x)]$ at $n+1$ isolated points corresponding to abscissae x_i $(i = 1, 2, \ldots, n+1)$, it is possible to construct "a curve of parabolic kind" $[p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$] which interpolates the fluent $[y(x_i) = p(x_i)$ for $i = 1, 2, \ldots, n+1]$. The area will subsequently be easy to calculate by approximation as the area of the surface subtended by the curve of parabolic kind (figure 8.20).

In Newton's words:

> *To square to a close approximation any curvilinear figure whatever, some number of whose ordinates can be ascertained.*
>
> Through the end-points of the ordinates draw a curve of parabolic kind with the aid of the preceding problems [the interpolation formulas of *Methodus Differentialis*]. For this will bound a figure which can always be squared, and whose area will be equal to the area of the figure proposed with close approximation.[87]

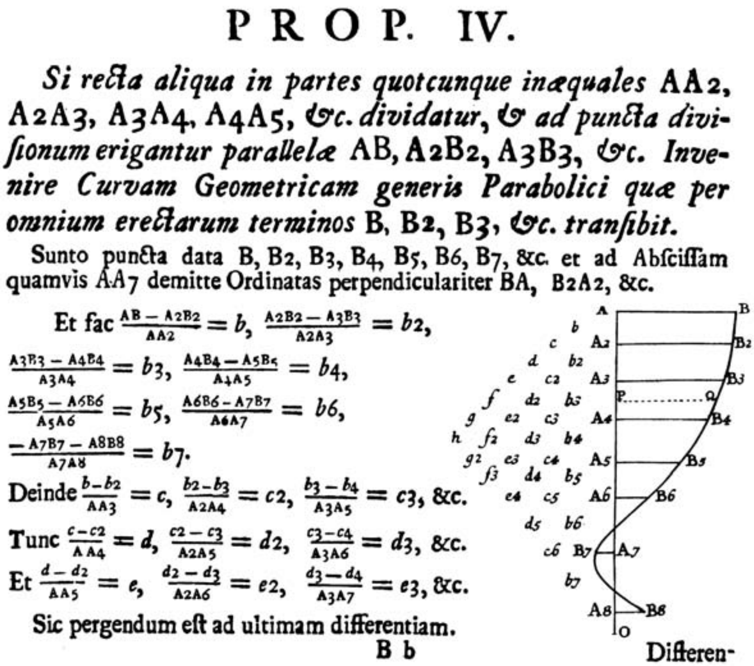
The Newton-Cotes formula originates from this research. Newton's work on interpolation dates from 1676 and was partly published in Lemma 5, Book 3, of the *Principia*; a full version appeared as *Methodus Differentialis* in the collection of mathematical essays edited by William Jones in 1711.[88]

---

[86] Add. 3964.4, f. 21r, and Add. 3965.14, ff. 611r–612v, in MP, 7, pp. 690–9 and 700–2.

[87] MP, 8, p. 253. "Figuram quamcunque Curvilineam quadrare quamproxime cujus Ordinatae aliquot inveniri possunt. Per terminos Ordinatarum ducatur linea Curva generis Parabolici ope Propositionum praecedentium. Haec enim figuram terminabit quae semper quadrari potest, et cujus Area aequabitur Areae figurae propositae quamproxime." Newton, *Analysis per Quantitatum* (1711), p. 100 = MP, 8, p. 252.

[88] Newton, *Analysis per Quantitatum* (1711), pp. 93–101. On *Methodus Differentialis*, see Fraser, *Newton's Interpolation Formulas* (1927) and Whiteside's commentary in MP, 4, pp. 36–51.



PROP. IV.

Si recta aliqua in partes quotcunque inæquales AA2, A2A3, A3A4, A4A5, *&c. dividatur, & ad puncta divisionum erigantur parallelæ* AB, A2B2, A3B3, *&c. Invenire Curvam Geometricam generis Parabolici quæ per omnium erectarum terminos* B, B2, B3, *&c. transibit.*

Sunto puncta data B, B2, B3, B4, B5, B6, B7, &c. et ad Absciffam quamvis AA7 demitte Ordinatas perpendiculariter BA, B2A2, &c.

Et fac $\frac{AB - A2B2}{AA2} = b$, $\frac{A2B2 - A3B3}{A2A3} = b2$,

$\frac{A3B3 - A4B4}{A3A4} = b3$, $\frac{A4B4 - A5B5}{A4A5} = b4$,

$\frac{A5B5 - A6B6}{A5A6} = b5$, $\frac{A6B6 - A7B7}{A6A7} = b6$,

$\frac{-A7B7 - A8B8}{A7A8} = b7$.

Deinde $\frac{b - b2}{AA3} = c$, $\frac{b2 - b3}{A2A4} = c2$, $\frac{b3 - b4}{A3A5} = c3$, &c.

Tunc $\frac{c - c2}{AA4} = d$, $\frac{c2 - c3}{A2A5} = d2$, $\frac{c3 - c4}{A3A6} = d3$, &c.

Et $\frac{d - d2}{AA5} = e$, $\frac{d2 - d3}{A2A6} = e2$, $\frac{d3 - d4}{A3A7} = e3$, &c.

Sic pergendum eft ad ultimam differentiam.

B b Differen-

Figure 8.20

Proposition 4 from *Methodus Differentialis* (1711). Here Newton sought a "geometrical curve of parabolic kind" passing through a finite (either even or odd) number n of points B, B_2, B_3, ... B_n. In the previous propositions the abscissae A, A_2, A_3, ... A_n were assumed to be equally spaced, a condition that is now done away with. In Propositions 5 and 6 Newton deployed interpolations in order to achieve approximate quadratures. Source: Newton, *Analysis per Quantitatum* (1711), p. 97. Courtesy of the Biblioteca Universitaria di Bologna.

8.7 A Question of Style

Here I consider some general characteristics of the early treatises on the analytical method that were discussed in chapters 7 and 8.

In the time span from 1666 to 1671, Newton produced some well-structured and carefully written treatises on the new analysis. The October 1666 tract on fluxions and even more so *De Analysi* and *De Methodis* have the form of publishable texts: they are addressed to readers. In these very early years Newton not only jotted down personal notes or results to be briefly communicated to peers. In the case of these treatises, he rather didactically and systematically elaborated very comprehensive treatments on series and fluxions. He did not assume that readers were particularly advanced in mathematics. Such a mature and didactic style is quite extraordinary for a young man and might be revealing of Newton's academic ambitions to become a suitable substitute for Barrow, the first Lucasian Professor.

Newton explicitly conceived his treatises as part of a genre that can broadly be associated with the British analytical school of Oughtred and Wallis. The analytical method was presented through a series of specific and increasingly difficult examples. In these early works Newton's method was not yet a theory but rather a panoply of techniques ultimately justified by their success in resolving problems concerning curvilinear figures. Most of these techniques had no firm foundation. The attempts to provide demonstrations that surfaced from time to time in Newton's work were far outnumbered by the folios in which the desire to show their effectiveness was given pride of place. An example is the use of power series, which is so important in the analytical method. Neither the binomial series nor the more elaborate methods for the resolution of affected equations were given a proper demonstration. These results were achieved via inductions, analogies, and extrapolations of Wallisian origin. The analytical parallelogram was nothing more than a paper tool explained by testing its successful functioning, that is, by placing asterisks and rulers associated with increasingly difficult polynomial equations. It was a graphic aid that allowed achieving fractional power series expansions whose convergence was to be verified by hand. The same algorithmic approach characterized the extraordinary variety of methods for squaring curves that Newton proposed in the long catalogues of *De Methodis*.

The man who so carefully and extensively elaborated such heuristic, pragmatically successful, yet ungrounded methods was the same natural philosopher who in 1670 wished to inject certainty into natural philosophy via the use of geometry. Early on in his career, roughly from the mid-1670s, he began to portray himself as an erudite Church historian and chronologist, a theologian and polyhistor whose style was modeled on late-Renaissance philology. Probably in a later period Newton began to mix his anti-Cartesianism with a strong conviction about the superiority of the pre-Aristotelian ancients over the moderns. These cultural orientations, destined to shape Newton's personality for years to come, increasingly distanced him from the analytical genre of his early treatises on the method of series and fluxions. This divergence between the style of Newton's early treatises on the new analysis and the style of his nonmathematical researches helps explain his interest in the analysis of the ancient geometers (see chapter 5) as well as his attempts to develop a synthetic version of the method of fluxions (see chapter 9).

9 The Synthetic Method of Fluxions

[The method] based on the genesis of surfaces by their motion of flow appears a more natural approach ... which will come to be still more perspicuous and resplendent if certain foundations are, as is customary with the synthetic method, first laid.

—Isaac Newton, 1671

The method of first and ultimate ratios which is set out near the beginning of the first book [of the *Principia*] in eleven lemmas is nothing other than a part of the method of fluxions ad moments synthetically demonstrated; and these lemmas are premised in order that with their benefit the following propositions found by means of the analytical method of fluxions and moments could be synthetically demonstrated. The elements of the method of fluxions and moments are given synthetic proof in Lemma II of Book 2.

—Isaac Newton, late 1710s

9.1 Synthetic Quadratures in *De Methodis*

9.1.1 Demonstrations with No Algebraic Calculation

The procedures considered in chapter 8 are analytical. But recall that, from Newton's point of view, analysis (resolution) did not provide a *solution*. After the resolutive, analytical stage, a geometrical construction or synthetic stage must follow. Thus, Newton developed a synthetic method of fluxions.

At the beginning of *De Methodis*, Newton somewhat parenthetically mentioned the need to provide a synthetic demonstration "from proper foundations" for the inverse method of fluxions. In a section devoted to the inverse problem (Problem 2), he wrote,

Epigraph sources: (1) MP, 3, pp. 283, 331. "sed magis naturalis videtur quae genesi superficierum ex fluendi motu innititur. ... quae magis perspicua et ornata evadet si fundamenta quaedam pro more methodi syntheticae praesternantur." MP, 3, pp. 282, 530. (2) MP, 8, p. 447. "Methodus rationum primarum et ultimarum quae sub initio libri primi in Lemmatibus XI exhibetur, nihil aliud est quam pars methodi fluxionum et momentorum synthetice demonstrata, & praemittuntur haec Lemmata ut eorum beneficio Propositiones sequentes per methodum analyticam fluxionum et momentorum inventae synthetice demonstrari possent. Elementa methodi fluxionum & momentorum demonstrantur synthetice in Lem. II Lib, II." MP, 8, p. 446.

> We have at last done with the problem [Problem 2] but its demonstration still remains. Not (in such mass of material) to digress too much in deriving one synthetically from proper foundations, it should be sufficient to indicate it briefly by analysis.[1]

Recall that the resolution of Problem 2 allows the determination of the areas of curvilinear figures, that is, the resolution of the problem of the quadrature of curves. In *De Methodis*, Newton developed several algebraic analytical methods of quadrature (§8.4).

After such a long and detailed analytical treatment of the problem of quadrature Newton proposed some synthetic constructions. He wrote,

> After the area of some curve has thus been found, careful consideration should be given to fabricating a demonstration of the construction which as far as permissible has no algebraic calculation, so that the theorem embellished with it may turn out worthy of public utterance. A general method of demonstration exists, indeed, and this I shall attempt to illustrate by the following examples.[2]

Newton's synthetic constructions and demonstrations from first principles are Barrovian in character: they resemble the theorems on quadratures found in Barrow's *Lectiones Geometricae*.

9.1.2 The Synthetic Quadrature of the Cissoid

Here I consider Newton's synthetic quadrature of the cissoid (Problem 9, Example 3, of *De Methodis*), whose analytical resolution was discussed in section 8.4.5, in order to provide a direct comparison with Barrow's procedures (§8.1.6). The problem is how to square the cissoid AeE, that is, how to determine the area of the surface $ACEeA$ (figure 9.1). Because of the defining property of the cissoid, CD, AC, and CE are in continued proportion, where ADQ is a circle. Newton proceeded as follows:

> *Proof of the construction of Example 3.* Let $DEe\partial$ be a moment of the surface $AHDEeA$ with $AD\partial A$ the contemporaneous moment of the segment $ADHA$. Draw

[1] MP, 3, p. 113. "Problema tandem confecimus sed demonstratio superest. Et in tanta rerum copia ne per nimias ambages e proprjis fundamentis Synthetice derivetur, sufficiat per Analysin sic breviter indicare." MP, 3, p. 112.

[2] MP, 3, p. 279. "Postquam Curvae alicujus area sic inventa fuerit; de constructionis demonstratione consulendum est, quacum sine Computo Algebraico quantum liceat contexta ornetur Theorema ut evadat publicae notitiae dignum. Estque demonstrandi methodus generalis quam sequentibus exemplis illustrare conabor." MP, 3, p. 278. I have slightly altered Whiteside's translation.

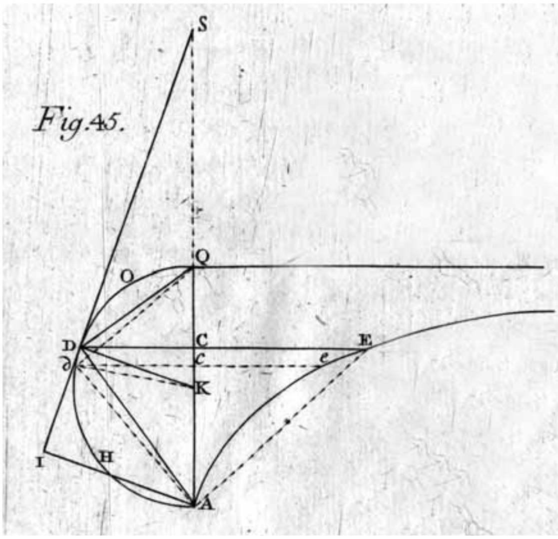


Figure 9.1
Cissoid AeE, from *De Methodis* (1671). Source: Newton, *Opuscula Mathematica, Philosophica et Philologica* (1744), 1, Tab. VII. Courtesy of the Biblioteca Angelo Mai (Bergamo).

the radius DK and let ∂e meet AQ in c, and there is[3]

$$Cc : D\partial = DC : DK.$$

Further, $DC : QA$ (or $2DK$) $= AC : DE$, so that[4]

$$Cc : 2D\partial \ (= DC : 2DK) = AC : DE,$$

and $Cc \times DE = 2D\partial \times AC$. Now to the moment $D\partial$ of the circumference extended in a straight line (that is, to the circle tangent) let fall the perpendicular AI, and AI will then be equal to AC, so that[5]

$$2D\partial \times AC \ (= 2D\partial \times AI) = 4 \text{ triangle } AD\partial.$$

[3] Here Newton deployed the similarity between the triangle DCK and the infinitesimal triangle whose sides are the "moment $D\partial$ of the circumference AD," the moment Cc of the abscissa AC, and the moment of the ordinate DC.
[4] This proportionality is a consequence of the defining property of the cissoid. In symbols: $DE = DC + CE = \sqrt{az - z^2} + z^2/\sqrt{az - z^2} = az/\sqrt{az - z^2} = QA \times AC : DC$, where (§8.4.5) $AQ = a$ and $AC = z$.
[5] The right triangles IDA and CDA are congruous; therefore $AI = AC$. Further, twice the area of the infinitesimal triangle $AD\partial$ is equal to its base $D\partial$ times its height AI.

> Hence 4 triangle $AD\partial = Cc \times DE =$ moment $DEe\partial$. Therefore each moment of the space $AHDEeA$ is four times the contemporaneous one of the segment $ADHA$, and consequently the total space is four times the total segment [i.e., $4\times$ segment $ADHA =$ area $AHDEeA$]. Q.E.D.[6]

The result coincides with the one achieved via algebraic fluxional analysis, yet it is only after such a geometrical construction, free of algebraic calculation, that Newton felt justified in writing "Q.E.D."

A few comments are in order. First, Newton here followed the canon of analysis and synthesis. In the synthesis he delivered a geometrical demonstration of a result that he had previously achieved via fluxional analysis. Second, Newton made it clear that such constructions and their demonstrations must be given by avoiding algebraic calculations. Third, only these geometrical constructions and demonstrations were, in his opinion, worthy of public utterance. Fourth, Newton's synthetic approach to quadratures was modeled upon Barrovian exemplars. Finally, in the synthetic quadratures Newton deployed moments (infinitesimal magnitudes).

On this last point, the equivalence between the area $AHDEeA$ and four times the area of segment $ADHA$ is proven by showing that the moment of $AHDEeA$ is four times the contemporary moment of $ADHA$, where a moment is the increase of the surface generated by the flow of the ordinates DE and AD acquired in an infinitesimal interval of time as the point on the circumference flows from D to ∂. In Newton's demonstration it was essential to regard the ratio of the rectilinear area $Cc \times DE$ to the curvilinear area $DEe\partial$ as a "ratio of equality," since "their difference … is infinitely less than they." Newton wrote,

> In demonstrations of this sort it should be observed that I take quantities as equal whose ratio is one of equality. And a ratio of equality is to be regarded as one which differs less from equality than any ratio of inequality [which] can possibly be assigned. Thus … I set the rectangle [$Cc \times DE$] equal to the space [$DEe\partial$] since (because their difference is infinitely less than they and so, in regard to them, zero) they have no ratio of inequality.[7]

Infinitesimals also occur when the moment $D\partial$ of the circumference is equated with a portion of the tangent.

As this example clearly shows, in writing *De Methodis*, Newton found himself in an uncomfortable position when seeking to abandon the heuristic analytical stage, in favor of demonstrative and "resplendent" constructions. The new analysis seemed to require a geometrical synthesis that was not contemplated by tradition, since infinitesimals occurred in its demonstrations. How could this new synthesis be justified? Newton gave this question serious attention.

[6] MP, 3, p. 281.
[7] MP, 3, p. 283. I have adapted this statement to Example 3.

Already in *De Methodis* there is a first attempt to formulate a reply:

> I have here used this method of proving that curves are equal or have a given ratio by means of the equality or given ratio of their moments since it has an affinity to the ones usually employed in these cases. However, that based on the genesis of surfaces by their motion of flow appears a more natural approach ... one which will come to be still more perspicuous and resplendent if certain foundations are, as is customary with the synthetic method, first laid.[8]

This "more natural approach" was first attempted in an Addendum to *De Methodis*, which Newton probably wrote in 1671; it was further elaborated in "Geometria Curvilinea," in Section 1, Book 1, of the *Principia*, and in *De Quadratura*.[9] In his mature years, Newton sometimes referred to this approach as the synthetic method of fluxions, as opposed to the analytical method of fluxions.[10]

9.2 "Geometria Curvilinea"

Newton perfected his attempts to formulate a synthetic method of fluxions in a treatise entitled "Geometria Curvilinea," which he composed in about 1680. In the Addendum to *De Methodis* he had already begun to develop definitions, axioms,

[8] MP, 3, pp. 283, 331. "Hac methodo probandi curvas per aequalitatem vel datam rationem momentorum aequales esse vel datam relationem habere hic usus sum quod cum methodis in his rebus usitatis affinitatem habeat; sed magis naturalis videtur quae genesi superficierum ex fluendi motu innititur. ... quae magis perspicua et ornata evadet si fundamenta quaedam pro more methodi syntheticae praesternantur." MP, 3, pp. 328, 329.

[9] The manuscript entitled by its editor "Addendum on the Theory of Geometrical Fluxions" is Add. 3960.4, ff. 33–46 = MP, 3, pp. 328–53. The "Geometria Curvilinea" is Add. 3963.7, f. 46r/3960.5, ff. 49–52/3963.7, ff. 48v–61v = MP, 4, pp. 420–85.

[10] Newton introduced this distinction when discussing the occurrence in the *Principia* of the analytical and synthetic methods of fluxions: "The synthetic method of fluxions occurs widespread in the following treatise [the *Principia*], and I have set its elements in the first eleven lemmas of the first book and in Lemma II of the second. Specimens of the analytical method occur in Proposition XLV and the Scholium to Proposition XCIII of Book 1, and in Propositions X and XIV of Book 2. It is, furthermore, described in the scholium to lemma II of Book 2. And from their composed demonstrations, also, the analysis by which the propositions were found out can be learnt by going backwards." MP, 8, pp. 455–7. Translation by Whiteside. Another interesting document is a loose catalogue entitled "De methodo fluxionum" inserted in a copy of the *Principia* that was in Newton's possession. "De methodo fluxionum. Lib. I. Sect. I. est de methodo rationum primarum et ultimarum ... haec est methodus momentorum synthetica. Eadem si Analytice tractetur evadit methodus momentorum Analytica, quam etiam methodum fluxionum voco. Lib. I. sect XIII Prop. 93. Schol. pag 202. Methodus solvendi Problemata per serie & momenta conjunctim exponitur. Lib. II. Lem. II. pag. 224. ostendo quomodo fluentium ex lateribus per multiplicatonem divisionem vel extractionem radicum genitarum momenta et fluxiones inveniri possunt. Lib. II. Prop. 19 [for 14]. pag 251 argumentum procedit per differentiam momentorum, ideoque ideam tunc habui momentorum secondorum, et primus hanc ideam in lucem edidi." Newton, *Principia*, pp. 793–4. See part IV for discussion of Newton's use of fluxions in the *Principia*.

and postulates concerning geometrical magnitudes varying by continuous flow.[11] As Newton observed in "Geometria Curvilinea" (where the axiomatization attempted in the Addendum was further elaborated), Euclid's *Elements* are "scarcely adequate for a work dealing with curves," his being the "foundations of the geometry of straight lines."[12] The *Elements* were not enough for Newton's purposes.

Newton observed that the moderns had attempted to trespass the boundaries of Euclidean geometry by adopting a new approach: in dealing with curvilinear figures they had introduced infinitesimals. Newton no longer wished to follow the moderns. In the opening of "Geometria Curvilinea," he chastised the "men of recent times" who, by uniting geometry and arithmetic, express themselves in an "intolerably roundabout way":

> Those who have taken the measure of curvilinear figures have usually viewed them as made up of infinitely many infinitely small parts; I, in fact, shall consider them as generated by growing, arguing that they are greater, equal or less according as they grow more swiftly, equally swiftly or more slowly from their beginning. And this swiftness of growth I shall call the fluxion of a quantity. So when a line is described by the movement of a point, the speed of the point—that is, the swiftness of the line's generation—will be its fluxion. I should have believed that this is the natural source for measuring quantities generated by continuous flow according to a precise law, both on account of the clarity and brevity of the reasoning involved and because of the simplicity of the conclusions and the illustrations required.[13]

The demonstrations in "Geometria Curvilinea" are not based on infinitesimals; rather, they depend on the determination of the limits of ratios and sums of vanishing magnitudes. Typically, Newton needed to evaluate the limit to which the ratio between two geometrical magnitudes tends when they vanish simultaneously. He therefore distanced himself from the use of infinitesimals that had surfaced in previous works of his like *De Analysis* and *De Methodis* (§8.3.5). After 1680, Newton consistently followed an approach based on limits rather than on infinitesimals, an approach that was nevertheless implicit in many of his early procedures, especially in the Addendum to *De Methodis*.

Newton systematically developed the limit approach, the method of first and ultimate ratios, in the *Principia* and *De Quadratura*. The following sections discuss

---

[11] For Newton's attempts at axiomatization of the geometry of flowing magnitudes, see MP, 3, p. 330, and MP, 4, pp. 424–8.

[12] MP, 4, p. 423.

[13] MP, 4, p. 423. "Qui curvilineas figuras dimensi sunt, eas tamquam ex partibus infinite parvis & multis constantes contemplari solent. Ego vero eas considerabo tanquam crescendo generatas, argumentatus eas majores, aequales, vel minores esse prout ab initio celerius, aeque celeriter, vel tardius crescunt. Et hanc crescendi celeritatem vocabo fluxionem quantitatis. Sic ubi linea describitur per motum puncti, velocitas puncti hoc est celeritas generationis lineae erit fluxio ejus. Genuinum hunc fontem esse mensurandi quantitates continuo fluxu juxta certam legem generatas crediderim, tum ob perspicuitatem, & brevitatem ratiocinij, tum ob simplicitatem conclusionum & schematum quae requiruntur." MP, 4, p. 422.

the first section of the *Principia*, where the method of first and ultimate ratios is presented (§9.3); Lemma 2, Book 2, of the *Principia*, where Newton published some propositions of "Geometria Curvilinea" (§9.4); and the beginning of *De Quadratura*, implementing first and ultimate ratios (§9.5).

9.3 First and Ultimate Ratios in the *Principia*

Section 1, Book 1, of the *Principia* is devoted to the method of first and ultimate ratios sketched in portions of *De Methodis* and in "Geometria Curvilinea."[14] In that Section, consisting of eleven Lemmas and a Scholium, there is a clear statement about the use of infinitesimals that resonates with "Geometria Curvilinea." Infinitesimals, Newton stated, are to be understood as a shorthand for "evanescent divisibles" (finite magnitudes tending to zero, as one would say today):

> [W]henever in what follows I consider quantities as consisting of particles or whenever I use curved line-elements in place of straight lines, I wish it always to be understood that I have in mind not indivisibles but evanescent divisibles, and not sums and ratios of definite parts but the limits of such sums and ratios, and that the force of such proofs always rests on the method of the preceding lemmas.[15]

Newton pointed out that the method of first and ultimate ratios rests on the following Lemma 1:

> Quantities, and also ratios of quantities, which in any finite time constantly tend to equality, and which before the end of that time approach so close to one another that their difference is less than any given quantity, become ultimately equal.

Newton's *ad absurdum* proof runs as follows:

> If you deny this, let them become ultimately unequal, and let their ultimate difference be D. Then they cannot approach so close to equality that their difference is less than the given difference D, contrary to the hypothesis.[16]

---

[14] On the method of first and ultimate ratios, see De Gandt, "Le Style Mathématique des *Principia* de Newton" (1986); Pourciau, "The Preliminary Mathematical Lemmas of Newton's *Principia*" (1998).

[15] Newton, *Principles*, pp. 441–2. "Proinde in sequentibus, si quando quantitates tanquam ex particulis constantes consideravero, vel si pro rectis usurpavero lineolas curvas; nolim indivisibilis, sed evanescentia divisibilis, non summas et rationes partium determinatarum, sed summarum et rationum limites semper intelligi; vimque talium demonstrationum ad methodum praecedentium lemmatum semper revocari." *Principia*, p. 87.

[16] Newton, *Principles*, p. 433. "Quantitates, ut et quantitatum rationes, quae ad aequalitatem tempore quovis finito constanter tendunt, et ante finem temporis illius propius ad invicem accedunt quam pro data quavis differentia, fiunt ultimo aequales. Si negas; fiunt ultimo inaequales, et sit earum ultima differentia D. Ergo nequeunt propius ad aequalitatem accedere quam pro data differentia D: contra hypothesin." *Principia*, p. 73.

To regard this principle as an anticipation of Cauchy's theory of limits would certainly be a mistake, since Newton's theory of limits is referred to a geometrical rather than a numerical model. The objects to which Newton applied his synthetic method of fluxions or method of first and ultimate ratios are geometrical quantities generated by continuous flow. A typical mathematical problem occurring in the *Principia* is the study of the limit to which the ratio of two geometrical fluents tends when they simultaneously vanish (Newton used the expression the "limit of the ratio of two vanishing quantities").

For instance, in Lemma 7, Newton proved that in a given curve (figure 9.2)

> the ultimate ratios of the arc [ACB], the chord [AB], and the tangent [AD] to one another are ratios of equality.[17]

To convey an idea of Newton's method of first and ultimate ratios, I consider his demonstration of Lemma 7 in detail.

This demonstration has the following structure. Consider two geometrical quantities X and Y that vanish simultaneously when points A and B come together. When the two quantities are finite, the ratio can in principle be determined by standard geometrical techniques. The problem is how to determine the limit of the ratio when B approaches point A. Newton constructed two other quantities, x and y, which always remain finite, so that $X/Y = x/y$. As B tends to A, the ratio X/Y tends to $0/0$, but the ratio x/y tends to a finite value, which is to be taken as the first or ultimate ratio of the vanishing quantities X and Y.

Here is Newton's proof of Lemma 7:

> For while point B approaches point A, let AB and AD be understood always to be produced to the distant points b and d; and let bd be drawn parallel to secant BD.

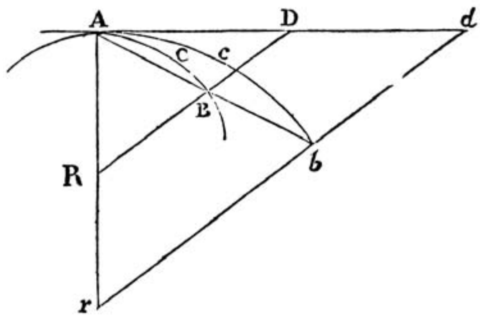


Figure 9.2

Limiting ratio of chord, tangent, and arc. Source: Newton, *Philosophiae Naturalis Principia Mathematica* (1726), p. 31. Courtesy of the Biblioteca Angelo Mai (Bergamo).

[17] Newton, *Principles*, p. 436.

> And let arc Acb be always similar to arc ACB. Then as points A and B come together, the angle dAb will vanish, by the preceding Lemma, and thus the straight lines Ab and Ad (which are always finite) and the intermediate arc Acb will coincide and therefore will be equal. Hence, the straight lines AB and AD and the intermediate arc ACB (which are always proportional to the lines Ab and Ad and the arc Acb respectively) will also vanish and will have to one another an ultimate ratio of equality.[18]

Further, in Lemma 2, Newton showed that the area of a curvilinear surface $AabcdE$ (figure 9.3) can be approached as the limit of the areas of rectilinear surfaces, inscribed $AKbLcMdD$ or circumscribed $AalbmcndoE$, as the number of component rectangles tends to infinity. Each rectilinear surface is composed of a finite number of rectangles with equal bases AB, BC, CD, etc. The proof—patterned on Barrow's example (§8.1.7)—is magisterial in its simplicity. Its structure can still be found in present-day calculus textbooks in the more general and abstract definition of the Riemann integral. It consists in showing that the difference between the areas of the circumscribed and inscribed figures tends to zero, as the number of rectangles is increased indefinitely. Indeed, this difference is equal to the area of rectangle $ABla$, which "because its width AB is diminished indefinitely, becomes less than any given rectangle."[19]

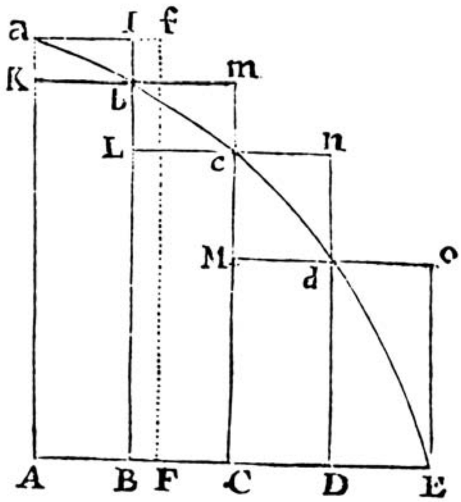


Figure 9.3

Approximating curvilinear areas. Source: Newton, *Philosophiae Naturalis Principia Mathematica* (1726), p. 28. Courtesy of the Biblioteca Angelo Mai (Bergamo).

[18] Newton, *Principles*, p. 436.
[19] Newton, *Principles*, p. 433.

In Lemma 2 and Lemma 7, Newton provided proofs of two assumptions that were customary in seventeenth-century geometrical practice, for it was often assumed that a curve can be conceived as a polygonal consisting of infinitely many infinitesimal sides (the tangent to a point being the prolongation of one of the infinitesimal sides) and that a curvilinear surface can be conceived as composed of infinitely many infinitesimal components. These assumptions can be seen at work in Newton's synthetic quadrature of the cissoid (§9.1). According to Newton, the method of first and ultimate ratios provided a foundation for such infinitesimal procedures. In "Geometria Curvilinea," the *Principia*, and *De Quadratura*, curves are smooth, and curvilinear surfaces are not seen as constituted by infinitesimal elemental surfaces.

The mathematical practice of the synthetic method, however, allowed the use of infinitesimals, since, Newton claimed, one could always reframe infinitesimalist proofs in terms of sums and ratios of vanishing quantities. The quadrature of the cissoid can stand as it is in *De Methodis*. The important thing here is that Newton's discourse in terms of moments has to be understood as being grounded on the theory of limits.

Since Newton downgraded infinitesimals, or moments, as heuristic tools that have to be grounded on limits, he needed to justify the limits themselves. In order to do so, Newton made use of geometrical and kinematical intuition. It is interesting to quote a pronouncement from Section 1 at some length:

> It may be objected that there is no such thing as an ultimate proportion of vanishing quantities, inasmuch as before vanishing the proportion is not ultimate, and after vanishing it does not exist at all. But by the same argument it could equally be contended that there is no ultimate velocity of a body reaching a certain place at which the motion ceases; for before the body arrives at this place, the velocity is not the ultimate velocity, and when it arrives there, there is no velocity at all. But the answer is easy; to understand the ultimate velocity as that with which a body is moving, neither before it arrives at its ultimate place and the motion ceases, nor after it has arrived there, but at the very instant when it arrives, that is, the very velocity with which the body arrives at its ultimate place and with which the motion ceases. And similarly the ultimate ratio of vanishing quantities is to be understood not as the ratio of quantities before they vanish or after they have vanished, but the ratio with which they vanish.[20]

[20] Newton, *Principles*, p. 442. "Objectio est quod quantitatum evanescentium nulla sit ultima proportio; quippe quae, antequam evanuerunt, non est ultima; ubi evanuerunt, nulla est. Sed et eodem argumento aeque contendi posset nullam esse corporis ad certum locum, ubi motus finiatur, pervenientis velocitatem ultimam: hand enim, antequam corpus attingit locum, non esse ultimam; ubi attingit, nullam esse. Et responsio facilis est: per velocitatem ultimam intelligi eam, quam corpus movetur, neque antequam attingit locum ultimum et motus cessat, neque postea, sed tunc cum attingit; id est, illam ipsam velocitatem quâcum corpus attingit locum ultimum et quâcum motus cessat. Et similiter per ultimam rationem quantitatum evanescentium, intelligendam esse rationem quantitatum, non antequam evanescunt, non postea, sed quâcum evanescunt." *Principia*,

As has been observed, the notion of mathematical magnitudes as generated by continuous flow in time provided two advantages that Newton greatly appreciated. The first is that the limiting procedures that are deployed in determining tangents and areas can be grounded on the continuity of motion, that is, it is possible to claim that the limits determined by such procedures exist and are unique because of the continuity of the generating motion. Further, the continuity observed in physical motions makes it possible to conceive of mathematics as a language applicable to the study of the natural world.

9.4 Lemma 2, Book 2, of the *Principia*

A more algorithmic approach to the direct method of fluxions can be found in Lemma 2, Book 2, of the *Principia*. This lemma, as Whiteside observed, occupies a somewhat incongruous position in the midst of Newton's treatment of projectile motion in resisting media. Its statement is used in the immediately subsequent Propositions 8 and 9, where the calculation of the moment of a square AP^2 is needed. This simple result was already deployed in Book 1. Whiteside documented the dependence of Lemma 2 upon "Geometria Curvilinea," some initial propositions of which it reproduced (Propositions 3–10), as well as the Addendum to *De Methodis*. Whiteside also surmised that Newton probably introduced this lemma as a reaction to Leibniz's recent publication (1684) of the differential calculus.[21] In the Scholium to Lemma 2, Newton mentioned his correspondence with Leibniz and the fact that the latter had developed a method "which hardly differed from mine except in the form of words and notations [a]and the concept of the generation of quantities[a]."[22]

In Lemma 2 Newton wrote,

> The moment of a generated quantity [*genitum*] is equal to the moments of each of the generating roots multiplied continually by the exponents of the powers of those roots and by their coefficients.[23]

The term genitum is here used to designate a fluent quantity increasing or decreasing by continual flux. It is a term that applies both to arithmetical and to geometrical quantities. As in *De Quadratura*, Newton aimed at a formulation that would encompass both algebraic and geometrical interpretation. A genitum, as one knows from "Geometria Curvilinea," should not be conceived as constituted by the addition of infinitesimal parts but as generated by motion:

---

p. 87.

[21] MP, 4, pp. 522–3, note 1.

[22] [aa]Added in the second edition of the *Principia* (1713). See Newton, *Principles*, p. 649.

[23] Newton, *Principles*, p. 646. "Momentum genitae aequatur momentis laterum singulorum generantium in eorundem laterum indices dignitatum & coefficientia continue ductis." *Principia*, p. 364.

> I call a *Genitum* every quantity that is, without addition or subtraction, generated from any roots or terms: in arithmetic by multiplication multiplication, division, or extraction of the root; in geometry by the finding either of products and roots or of extreme and mean proportionals. Quantities of this sorts are products, quotients, roots, rectangles, squares, cubes, square roots, cube roots, and the like. I here consider these quantities as indeterminate and variable, and increasing or decreasing as if by a continual motion or flux; and it is their instantaneous increments or decrements that I mean by the word "moments," in such a way that increments are considered as added or positive moments, and decrements as subtracted and negative moments. But take care: do not understand them as finite particles! [a]Finite particles are not moments, but the very quantity generated from the moments.[a] They must be understood to be the just-now nascent beginning of finite magnitudes. For in this lemma the magnitude of moments is not regarded, but only their first proportion when nascent.[24]

In his definition of moment Newton seems to echo what Wallis had written in *A Defense of the Treatise of the Angle of Contact*, where one finds indivisibles defined as inchoative or inceptive quantities, generative of finite magnitudes (§7.2).

Newton next stated several rules for the direct algorithm of fluxions. Whiteside's guess that Lemma 2 was a response to Leibniz seems confirmed not only by Newton's guarded criticism of the use of infinitesimals (which seems to characterize Leibniz's first publication on the differential calculus) but also by the algorithmic character of these rules. In works like *De Analysi* and *De Methodis*, Newton had presented the direct algorithm via application to a number of chosen, increasingly difficult examples of fluents (§8.3). Here he aimed at generalizing the rules of the direct method of fluxions in a way reminiscent of what Leibniz had achieved in "Nova Methodus":

> Therefore the meaning of this Lemma is that if the moments of any quantities A, B, C, ... increasing or decreasing by a continual motion, or the velocities of mutation which are proportional to these moments are called a, b, c, ... then the moment or mutation of the generated rectangle AB would be $aB + bA$, and the moment of the generated solid ABC would be $aBC + bAC + cAB$, and the moments of

[24] Newton, *Principles*, p. 647. [aa] is a variant introduced in the second (1713) edition. "Genitam voco quantitatem omnem, quae ex lateribus vel terminis quibuscunque in arithmetica per multiplicationem, divisionem & extractionem radicum; in geometria per inventionem vel contentorum & laterum, vel extremarum & mediarum proportionalium, sine additione & subductione generatur. Ejusmodi quantitates sunt facti, quoti, radices, rectangula, quadrata, cubi, latera quadrata, latera cubica, & similes. Has quantitates, ut indeterminatas & instabiles, & quasi motu fluxuve perpetuo crescentes vel decrescentes, hic considero; & earum incrementa vel decrementa momentanea sum nomine momentorum intelligo: ita ut incrementa pro momentis addititiis seu affirmativis, ac decrementa pro subductitiis seu negativis habeantur. Cave tamen intellexeris particulas finitas. Particulae finitae non sunt momenta, sed quantitates ipsæ ex momentis genitae. Intelligenda sunt principia jamjam nascentia finitarum magnitudinum. Neque enim spectatur in hoc lemmate magnitudo momentorum, sed prima nascentium proportio." *Principia*, pp. 364–5.

> the generated powers A^2, A^3, A^4, $A^{\frac{1}{2}}$, $A^{\frac{3}{2}}$, $A^{\frac{1}{3}}$, $A^{\frac{2}{3}}$, A^{-1}, A^{-2}, and $A^{-\frac{1}{2}}$ would be $2aA$, $3aA^2$, $4aA^3$, $\frac{1}{2}aA^{-\frac{1}{2}}$, $\frac{3}{2}aA^{\frac{1}{2}}$, $\frac{1}{3}aA^{-\frac{2}{3}}$, $\frac{2}{3}aA^{-\frac{1}{3}}$, $-aA^{-2}$, $-2aA^{-3}$, and $-\frac{1}{2}aA^{-\frac{3}{2}}$ respectively. And generally, the moment of any power $A^{\frac{n}{m}}$ would be $\dfrac{n}{m}aA^{\frac{n-m}{m}}$. Likewise, the moment of the generated quantity A^2B would be $2aAB + bA^2$, and the moment of the generated quantity $A^3B^4C^2$ would be $3aA^2B^4C^2 + 4bA^3B^3C^2 + 2cA^3B^4C$, and the moment of the generated quantity $\dfrac{A^3}{B^2}$ or A^3B^{-2} would be $3aA^2B^{-2} - 2bA^3B^{-3}$, and so on.[25]

In the following demonstration of Case 1 of Lemma 2, Newton made a famous blunder. In his attempt to prove the rule for the product AB he stumbled into a "momentous" mistake. It is interesting to note that neither in "Geometria Curvilinea" nor in the Addendum to *De Methodis* had he made this mistake (the faulty reasoning had actually been crossed out).[26] He wrote,

> CASE 1. Any rectangle as AB increased by a continual motion, when the halves of the moments $\frac{1}{2}a$ and $\frac{1}{2}b$, were lacking from the sides A and B, was $A - \frac{1}{2}a$ multiplied by $B - \frac{1}{2}b$, or $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$, and as soon as the sides A and B have been increased by the other halves of the moments; it comes out $A + \frac{1}{2}a$ multiplied by $B + \frac{1}{2}b$, or $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$. Subtract the former rectangle from this rectangle, and there will remain the excess $aB + bA$. Therefore by the total increments a and b of the sides, there is generated the increment $aB + bA$ of the rectangle. *Q.E.D.*[27]

In the remaining Cases 2–6, Newton proved that the moment of A^n is naA^{n-1}, that of A^{-n} is $\dfrac{-na}{A^{n+1}}$, that of $A^{\frac{m}{n}}$ is $\dfrac{m}{n}aA^{\frac{m-n}{n}}$, and that of A^mB^n is $maA^{m-1}B^n + nbB^{n-1}A^m$. All these results are algorithmic and stated with a generality that cannot be found in the treatment of Problem 1 of *De Methodis*.

In one of the interleaved copies of the *Principia* that Newton kept in his later years he referred to Lemma 2 as "the foundation of a general method of which I wrote [in 1671]"; he also argued that Lemma 2 "demonstrates synthetically" what was "explained analytically" in *De Methodis*.[28] But since the demonstrations of all the cases considered in Lemma 2 depend upon the faulty demonstration of Case 1, Newton's achievement cannot be said to have been satisfactory. Lemma 2 is nevertheless a notable attempt to summarize the rules of the direct method of fluxions in a compact and general form.

---

[25] Newton, *Principles*, pp. 647–8.
[26] MP, 3, pp. 330–5.
[27] Newton, *Principles*, p. 648.
[28] This note was probably written in 1714. MP, 3, pp. 525–6.

Lemma 2 is completed by a Scholium, in which Newton referred to Leibniz's differential calculus. This Scholium played a significant role in Newton's dispute with Leibniz (see part VI).

9.5 Limits in *De Quadratura*

The method of first and ultimate ratios developed in "Geometria Curvilinea" and in Section 1, Book 1, of the *Principia*, was also adopted in *De Quadratura*. When Newton was preparing *De Quadratura* for publication as an appendix to the *Opticks* (1704), he prefaced it with an introduction in which he gave a mature statement of the method of first and ultimate ratios. These lines are often quoted in the literature, particularly the pompous opening:

> Mathematical quantities I here consider not as consisting of least possible parts, but as described by a continuous motion. ... These geneses take place in the reality of physical nature and are daily witnessed in the motion of bodies. And in much this manner the ancients, by drawing mobile straight lines into the length of stationary ones, taught the genesis of rectangles.[29]

Here are summarized a few of Newton's preferred ideas concerning the foundations of the method of fluxions. Newton stated that he was not using infinitesimals but rather conceiving magnitudes (fluents) as generated by continuous motion. He added that mathematical fluents are analogous to real motions occurring *in rerum natura* and that his method was akin to the ones employed by ancient authorities. But while the ancients had used *ad absurdum* reasoning in order to prove propositions concerning curvilinear areas or volumes, Newton adopted his handier method of limits.

De Quadratura is a highly symbolic work. Newton, however, claimed that algebraic symbols should be understood, whenever possible, in terms of finite magnitudes "visible to the eye" rather than in terms of infinitesimals. Indeed, in the method of first and ultimate ratios, finite geometrical magnitudes are associated with vanishing magnitudes:

> For fluxions are finite quantities and real, and consequently ought to have their own symbols; and each time it can conveniently so be done, it is preferable to express them by finite lines visible to the eye rather than by infinitely small ones.[30]

---

[29] MP, 8, p. 123. "Quantitates Mathematicas non ut ex partibus quam minimis constantes sed ut motu continuo descriptas hic considero ... Hae Geneses in rerum natura locum vere habent & in motu corporum quotidie cernuntur. Et ad hunc modum Veteres ducendo rectas mobiles in longitudinem rectaurum immobilium genesin docuerunt rectangulorum." MP, 8, p. 122.

[30] MP, 8, pp. 113, 115. "Nam fluxiones sunt quantitates finitae et verae ideoque symbola sua habere debent, et quoties commode fieri potest praestat ipsas per lineas finitas coram oculis exponere quam per infinite parvas." MP, 8, pp. 112, 114.

In illustrating his method of limits Newton presented an algebraic case, after having discussed some geometrical limit procedures similar to the ones he had employed in "Geometria Curvilinea" and the *Principia.* Newton's (strikingly modern) algebraic case is as follows:

> *Let the quantity x flow uniformly and the fluxion of the quantity x^n needs to be found* [Note that in Newton's notation n can be fractional]. In the time that the quantity x comes in its flux to be $x+o$, the quantity x^n will come to be $(x+o)^n$, that is [when expanded] by the method of infinite series
>
> $$x^n + nox^{n-1} + \frac{1}{2}(n^2-n)o^2x^{n-2} + \cdots; \tag{9.1}$$
>
> and so the augments o and $nox^{n-1} + \frac{1}{2}(n^2-n)o^2x^{n-2} + \cdots$ are one to the other as 1 and $nx^{n-1} + \frac{1}{2}(n^2-n)ox^{n-2} + \cdots$. Now let those augments come to vanish and their last ratio will be 1 to nx^{n-1}; consequently the fluxion of the quantity x is to the fluxion of the quantity x^n as 1 to nx^{n-1}.[31]

Note that the increment o is finite and that the calculation aims at determining the limit of the ratio $o/((x+o)^n - x^n)$ as o tends to zero. The limit is the one attained *precisely* when the two quantities vanish simultaneously; it is not a limit calculated when the two quantities differ from zero by an infinitesimal quantity. Such reasoning, according to Newton, would be faulty because it would introduce errors in mathematics: "The most minute errors are not in mathematical matters to be scorned."[32]

In the lines immediately following this demonstration, Newton stated that such limit procedures are in harmony with the geometry of the ancients:

> In finite quantities, however, to institute analysis in this way and to investigate the first or last ratios of nascent or vanishing finites is in harmony with the geometry of the ancients, and I wanted to show that in the method of fluxions there should be no need to introduce infinitely small figures into geometry.[33]

It is instructive to consider the geometrical limit procedures that Newton proposed besides the algebraic one. The first has to do with the determination of tangents to plane curves; it is a limit procedure that expresses the argument in terms of the momentary increases found in *De Methodis* (§8.3.6). The second is reminiscent of many geometrical limit procedures characteristic of "Geometria Curvilinea" and the *Principia.*

---

[31] MP, 8, pp. 126–9.

[32] MP, 8, p. 125.

[33] MP, 8, p. 129. "In finitis autem qantitatibus Analysin sic instituere, et finitarum nascentium vel evanescentium rationes primas vel ultimas investigare, consonum est Geometriae Veterum: et volui ostendere quod in Methodo Fluxionum non opus sit figuras infinite parvas in Geometriam introducere." MP, 8, p. 128.

In his first example Newton justifies the basic formula for the subtangent ($VB = y\dot{x}/\dot{y}$) that was widely employed in *De Methodis* in terms of limits (figure 9.4):[34]

> Draw the straight line Cc and produce it to K. Let the ordinate bc go back into its former place BC and, as the points c and C come together, the straight line CK will coincide with the tangent CH, and so the vanishingly small triangle CEc will as it attains its last form end up akin to the triangle CET and its vanishing sides CE, Ec and Cc will ultimately be to one another as the sides CE, ET and CT of the other triangle CET: in this proportion in consequence are the fluxions of the lines AB, BC and AC. If the points C and c are at any small distance apart from each other, the straight line CK will be a small distance away from the tangent CH; in order that the line CK shall coincide with the tangent CH and so the last ratios of the lines CE, Ec and Cc be discovered, the points C and c must come together and entirely coincide. The most minute errors are not in mathematical matters to be scorned.[35]

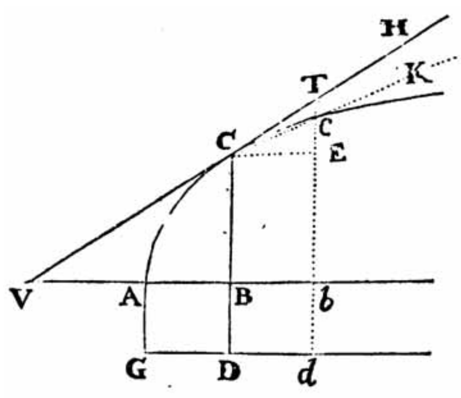


Figure 9.4
Determination of tangents in *De Quadratura*. Source: Newton, *Analysis per Quantitatum* (1711), p. 42. Courtesy of the Biblioteca Universitaria di Bologna.

[34] The subtangent is defined as the segment of the axis of abscissae lying between the abscissa of the point at which a tangent is drawn to a curve (in figure 9.4, B) and the intercept of the tangent with the axis of abscissae (in figure 9.4, V).

[35] MP, 8, p. 125. "Agatur recta Cc & producatur eadem at K. Redeat ordinata bc in locum suum priorem BC, & coeuntibus puctis C & c, recta CK coincidet cum tangente CH, & triangulum evanescens CEc in ultima sua forma evadet simile triangulo CET, & ejus latera evanescentia CE, Ec & Cc erunt inter se ut sunt trianguli alterius CET latera CE, ET & CT, & propterea in hac ratione sunt fluxiones linearum AB, BC & AC. Si puncta C & c parvo quovis intervallo ab invicem distant recta CK parvo intervallo a tangente CH distabit. Ut recta CK cum tangente CH coincidat & rationes ultimae linearum CE, Ec & Cc inveniantur, debent puncta C & c coire & omnino coincidere. Errores quam minimi in rebus Mathematicis non sunt contemnendi." MP, 8, p. 124.

Note that the increments of the abscissa and ordinate are finite and that Newton's procedure aims at the determination of the limit of the ratio of the vanishing increments as the points C and c coincide. His claim is that by following this limit procedure no errors are introduced in mathematics.

Newton's other example is equally instructive (figure 9.5):

> *The straight line PB revolving round the given pole P shall intersect another straight line AB given in position: there is required the ratio of the fluxions of those lines AB and PB.*
>
> Let the line PB advance from its place PB into the new position Pb; in Pb take PC equal to PB, and to AB draw PD such that the angle $\widehat{bPD}$ is equal to $\widehat{bBC}$: then, because the triangles bBC and bPD are similar, the augment Bb will be to the augment Cb as Pb to Db.[36] Now let Pb return to its former place PB so that those [finite] augments shall come to vanish, and their last [ultimate] ratio as they do so—the last ratio of Pb to Db, that is—will be that had by PB to DB where the angle $\widehat{PDB}$ is right; and in this ratio, accordingly, is the fluxion of AB to the fluxion of PB.[37]

This is a simple example of a Newtonian geometrical limit procedure typical of "Geometria Curvilinea" and the *Principia*, whereby the ratio of the finite lines "visible to the eye" (PB/DB) represents the limit of the ratio of the vanishing augments (Bb/Cb).[38]

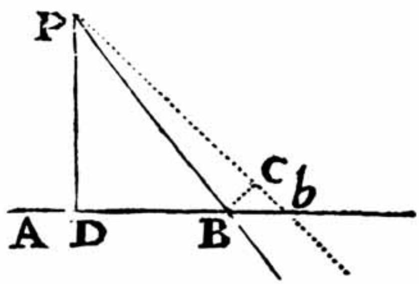


Figure 9.5

Geometrical limits in *De Quadratura*. Source: Newton, *Analysis per Quantitatum* (1711), p. 43. Courtesy of the Biblioteca Universitaria di Bologna.

---

[36] Newton established an identity between the variable ratio (Bb/Cb) of the two finite augments, Bb and Cb, which simultaneously tend to zero when Pb returns to its former place PB, and a variable ratio (Pb/Db) between two magnitudes, Pb and Db, which remain finite throughout the limiting process.

[37] MP, 8, p. 127.

[38] The limit procedure is based on the method that De Gandt has termed the "method of finite witnesses." De Gandt, "Le Style Mathématique des *Principia* de Newton" (1986).

9.6 A Method Worthy of Public Utterance

After having developed his new analysis (the analytical method of series and fluxions) Newton turned to synthesis: the geometrical construction of the resolutions attained by fluxional analysis. He first performed these constructions in Barrovian terms by deploying moments, the infinitely small augments by which fluent quantities increase during each infinitely small interval of time. Dissatisfied with this technique of the moderns, Newton justified the use of moments as a shorthand for limiting procedures codified by the method of first and ultimate ratios. He explicitly drew a comparison between these limiting procedures and the ancient geometers' method of exhaustion. The latter, however, was based on long and cumbersome indirect *ad absurdum* proofs. Newton proposed his method of limits as a more direct way of avoiding infinitesimals. In his mathematical practice, he seldom used limit arguments but rather handled infinitesimal magnitudes. Newton nevertheless warned the reader that infinitesimals should always be understood as finite vanishing magnitudes, and that some of the typical assumptions of geometrical reasoning based on infinitesimals were grounded in basic lemmas such as those occurring in the opening section of the *Principia*.[39] In doing so, Newton endorsed a well-established pattern of justification of infinitesimal techniques, a pattern followed by mathematicians like Wallis. While Wallis had often affirmed the equivalence between his arithmetic of infinities and exhaustion techniques, he was eager to divulge his new heuristic method and attributed little importance to any attempt to fully reformulate new analysis according to the ancient style (§7.2). By contrast, Newton's reformulation of his analytical method of fluxions in synthetic terms derived from a profound conviction that only synthetic constructions were "worthy of public utterance," the modern analysis proving inadequate from the point of view of the standards of certainty set by ancient tradition and indeed Newton's program for a mathematicized natural philosophy.

Newton's method of limits has aroused a great interest in the secondary literature. It is possible to identify a first wave of commentaries on the cogency of first and ultimate ratios in the polemical pamphlets surrounding the publication of George Berkeley's *Analyst* (1734). Recently, there has been a revival of interest in this

[39] "I have presented these lemmas before the propositions in order to avoid the tedium of working out lengthy proofs by *reductio ad absurdum* in the manner of the ancient geometers. Indeed, proofs are rendered more concise by the method of indivisibles. But since the hypothesis of infinitesimals is rather harsh and this method is therefore accounted less geometrical, I have preferred to make the proofs of what follows depend on the ultimate sums and ratios of vanishing quantities and the first sums and ratios of nascent quantities, that is, on the limits of such sums and ratios, and therefore to present proofs of those limits beforehand as briefly as I could. For the same result is obtained by these as by the method of indivisibles, and we shall be on safer ground by using principles that have been proved." Newton, *Principles*, p. 441.

topic. In this chapter I did not address the complex and sophisticated issues that have emerged, especially in this last decade of Newtonian studies. Partly this is due to my desire to avoid repeating the kind of work carried out by other scholars. On a more fundamental level, I am reluctant to enter the debate on "the limits of Newton's limits" because I fear it is based on an agenda foreign to that so passionately endorsed by the author of the *Principia*. Newton never doubted the logical rigor or cogency of his mathematical procedures. He showed no doubts about the results he achieved via the binomial theorem, the resolution of affected equations, the analytical parallelogram, or the catalogues of curves. Newton's concern with method was not polarized by questions of definition of basic terms and cogency of deductive rules, questions that typically emerge when one views mathematics as consisting of formalized axiomatic theories. Rather than being worried by questions of definition (concerning, say, continuity, tangency, or curvature) and convergence—which he introduced in a very intuitive way—Newton took pains to reformulate the results he had achieved algebraically in a geometrical language more compatible with the venerated ancient tradition. Newton's discourse concerning the method of limits is heavily influenced by his concern with *resolutio* and *compositio*, the comparison between the modern algebraic analytical path and ancient porismatic analysis, and the style and genre of his own mathematical work. In other words, Newton's interest in mathematical method was not driven by foundational questions concerning rigor but by stylistic questions concerning elegance, antimodernism, compatibility with the ancient tradition, and the visualization of mathematical concepts. As Newton put it, the synthetic method was preferable because of "clarity and brevity of the reasoning involved and because of the simplicity of the conclusions and the illustrations required."[40]

When present-day mathematicians, well trained in the study of calculus in terms of modern convergence theory, devote their critical attention to Newton's theory of limits, they often reach different conclusions. Newton's demonstrations in terms of limits are always very succinct. To the modern reader they seem in need of some sort of explication. One can, therefore, attempt to use modern symbolism and set the original text straight in a cogent algebraic form. In a way, this choice goes exactly against Newton's *desiderata*; it is a choice that generally leads to a disastrous distortion of the original. A better policy is that of providing the necessary definitions and convergence arguments in Newton's own geometrical terms by making use of Newton's published works or manuscripts. Those who follow this strategy attempt to piece several Newtonian *ipsissima verba* together in the hope of reconstructing a logically rigorous argument. The results are more interesting in this case but often so convoluted that I fear Newton himself would have been surprised by such lengthy detours. Newton's preference for geometry was determined, one

[40] MP, 4, p. 423.

should recall, by the fact that he believed it excelled in conciseness and elegance over algebra.

Ultimately, many approaches are possible. Hence, debates and quarrels between detractors and supporters of Newton are rather frequent. Some admirers of Newton view his geometrical limit arguments as extraordinary in their simplicity. There are mathematicians who, somewhat dissatisfied by the algebraic style that is prevalent in certain calculus textbooks (or, rather, was prevalent in the Bourbakist era), find the reading of Newtonian limit arguments refreshing and inspiring. Others have noted that such geometrical elegance is a viable choice for tackling only a handful of isolated and rather elementary problems. Both viewpoints, that of the enthusiasts and that of the skeptics, capture a part of truth. Newton's geometric limit arguments, epitomized in the *Principia*, are esthetically rewarding. However, when Newton tried to use the method of first and ultimate ratios to deal with more advanced problems, he incurred difficulties mainly due to the lack of a systematic way of controlling higher-order infinitesimals in geometrical terms. As Newton knew very well, some geometrical vanishing magnitudes can be discarded in comparison with other vanishing magnitudes. This happens, of course, when their ultimate ratio does not tend to a finite value.[41] Establishing when ratios of vanishing magnitudes have a finite value in more complicated geometrical diagrams can become, and did become in the more advanced propositions of the *Principia*, a real nightmare.

How successful was Newton in reformulating the analytical method of fluxions in a synthetic style more attuned to his philosophical agenda? As the discussion in part IV shows, Newton used the synthetic style in the *Principia* in a very successful way indeed. Particularly in the first sections of his *magnum opus*, Newton either employed his peculiar geometry of vanishing magnitudes or directly handled geometrical infinitesimal magnitudes. The synthetic method, however, did not allow Newton to achieve *all* the results he needed for developing gravitation theory. In more advanced sections of Book 1, and in most propositions of Book 3, Newton was forced to use, if somewhat obliquely, highly algebraic methods (most notably the analytical methods of quadrature treated in *De Methodis*) that were not rendered explicit in the printed text.

[41] For instance, DB in figure 9.2 vanishes more rapidly than AB, since the ratio DB/AB tends to zero when points B and A come together. In Lemma 11, Section 1, of the *Principia*, Newton proved that the vanishing magnitude DB is as the square of the vanishing magnitude AB, since the ratio AB^2/DB tends to the length of a chord of the osculating circle. Newton, *Principles*, pp. 439–40.

IV Natural Philosophy

Part VI considers some of the features of the Newtonian mathematization of force and motion. Its main focus is the *Principia*. The time span is from 1684 to 1687 (the period in which the elaboration of Newton's *magnum opus* took place), but some considerations regarding later additions, most notably the second edition (1713), are also be included.

My aim is to illustrate how the canon of problem resolution and composition that Newton endorsed in his previous works and his policy of publication shaped his most famous book.

The mathematical structure of the *Principia* is rather complex. A variety of geometrical methods play a prominent role. Newton's ability to obtain profound results in natural philosophy by purely geometrical means is astonishing. The *Principia* is still an inspiring book for the practicing mathematical physicist. Yet in this work Newton also employed algebra and calculus (albeit not systematically). The attempt to understand how the various components of Newton's mathematical methods interact in the *Principia* proves a difficult historiographic exercise.

In chapter 10, I present a survey of the *Principia* and address the *vexata quaestio* of the extent to which Newton relied on algebra and calculus in composing the work. An answer is provided in the following two chapters, discussing how Newton used the common analysis (chapter 11) and new analysis (chapter 12) in order to resolve three specific problems of fundamental importance for his theory of gravitation.

10 The *Principia*

> The ponderous instrument of synthesis, so effective in [Newton's] hands, has never since been grasped by one who could use it for such purposes; and we gaze at it with admiring curiosity, as on some gigantic implement of war, which stands idle among the memorials of ancient days, and makes us wonder what manner of man he was who could wield as a weapon what we can hardly lift as a burden.
>
> —William Whewell, 1837

10.1 Genesis of the *Principia*

10.1.1 Initial Influences

Newton's earliest studies on the laws of motions and gravity took place in late 1664 and early 1665.[1] As in the case of mathematics, his starting point was Descartes. Newton commented upon Part 2 of Descartes' *Principia Philosophiae* (1644) with particular insight. It is believed that the title of Newton's *magnum opus* was conceived as a criticism of the French philosopher, whose work, Newton thought, lacked adequate mathematical principles. From Descartes, Newton learned about the law of inertia, which was to become the first axiom or law of motion of the *Principia*: a body is at rest or moves in a straight line with constant speed until a force is applied to it. Unaccelerated rectilinear motion is the condition in which a body naturally perseveres; it does not need, as it was thought in the Aristotelian tradition, a mover.

By the early 1660s natural philosophers had examined two cases of accelerated motion: rectilinear uniformly accelerated motion and uniform circular motion. The first case occurs in the study of bodies falling close to the earth's surface. As Galileo taught (Newton knew his *Dialogo* through Salusbury's translation), these bodies describe parabolic trajectories by composition of inertial and uniformly accelerated motions.[2]

From the very beginning of his studies, Newton tried to subject the motions of bodies to mathematical laws. His first mathematical law in this field, which he achieved in work carried out in the years 1664–1668, is nowadays attributed to Christiaan Huygens, since it was first published in 1673 in the appendix to

---

Epigraph from Whewell, *History of the Inductive Sciences* (1837), p. 167.

[1] Newton, *Certain Philosophical Questions* (1983), pp. 275–309.

[2] De Gandt, *Force and Geometry* (1995), pp. 117–39.

the latter's *Horologium Oscillatorium.* In modern terms, the law states that the centripetal acceleration of a body that moves in a circular trajectory with constant speed is proportional to the square of the speed and inversely proportional to the radius.

Early in his youthful studies Newton intuited how to generalize mathematical results on uniform circular motion to more general cases. In late 1664 or early 1665, he observed in his Waste Book,

> If ye body *b* moved in an Ellipsis, yn its force in each point (if its motion in yt point bee given) will bee found by a tangent circle of equall crookednesse wth yt point of ye Ellipsis.[3]

This proved an extremely fertile insight. How can one go beyond the simple cases of uniformly accelerated rectilinear and circular uniform motion? In order to estimate the acceleration, for instance, in an elliptical trajectory (like planetary orbits) one can assume that locally the body moves with circular uniform motion along the osculating circle. Recall that in those years Newton developed fluxional techniques to calculate the radius of curvature of plane curves. He later exploited this early intuition. In his *Principia*, and particularly in the second edition (1713), Newton made use of the fact that the instantaneous normal acceleration a_N in a non circular orbit can be calculated by locally applying Huygens's laws for circular uniform motion: in modern terms, $|a_N| = v^2/\rho$ (v instantaneous speed, ρ radius of curvature).[4]

What about Newton's early thoughts on planetary motions? The few extant records indicate that for many years he remained trapped in the framework of Cartesian vortex theory. There are reasons to believe that, in the 1660s, Newton believed that the planets orbit the sun because they are transferred by a vortex in nearly circular orbits. It seems likely that at this early stage Newton followed Descartes in assuming that the circular motion of the planet generates an effort to recede from the center, a centrifugal conatus that would accelerate the planet away from the sun. The centrifugal conatus would be counterbalanced by the centripetal action of the vortex, which thus keeps the planet at an approximately constant distance from the sun. Assuming that the orbits are perfectly circular, it was all too easy, by combining Huygens's law with Kepler's third law, to verify that the planets' radial (centrifugal) acceleration varies inversely as the square of their distance from the sun. This inverse-square law was thus attained in a context far removed from gravitation theory.[5]

---

[3] Add. 4004, f. 1r. MP, 1, p. 456.

[4] Brackenridge and Nauenberg, "Curvature in Newton's Dynamics" (2002).

[5] Newton's early manuscripts on circular motion and planetary motion have been often examined. The main sources concern the study of uniform circular motion in the Waste Book (Add. 4004), and more refined analyses (the comparison between gravity and centrifugal force on the earth's surface, the use of the laws of Huygens and Kepler to conclude that the planets' "endevors to recede

10.1.2 The Role of Hooke

Instrumental to Newton's shift from these Cartesian views was Robert Hooke. Hooke in 1679 attempted to revive his correspondence with Newton, which had been interrupted in the aftermath of the polemic on the *experimentum crucis* (see chapter 2). Hooke, recently appointed secretary of the Royal Society, told Newton of a new hypothesis of his according to which planets, moving in a space void of resistance, describe orbits around the sun because of a rectilinear inertial motion by the tangent and an attractive motion toward the sun. Hooke thus disposed both of the Cartesian endeavor to recede from the center and of the Cartesian vortex. Only one *centripetal* force directed toward the sun, he argued, is needed to deviate the unresisted inertial rectilinear motion of the planets. Newton soon discovered that Hooke's hypothesis was mathematically fruitful. The most fruitful insight he achieved (it is unclear exactly when) was that a body moving in a space void of resistance and attracted by a central force must obey Kepler's area law and that, vice versa, a body that moves in accordance to the area law must be accelerated by a central force.[6]

It is difficult to reconstruct the steps that led Newton to conceive of universal gravitation. Certainly, Hooke's contribution was momentous; historians are now reevaluating the role of the Royal Society's secretary in formulating the new cosmology. Hooke's ideas were certainly revolutionary, and the extant records prove that Newton did not immediately endorse them. Newton continued to believe that the planetary motions were caused by revolving ether. Contrary to Descartes, he seems to have interpreted this ethereal medium in nonmechanistic terms somewhat reminiscent of his alchemical researches. Until 1681, Newton discussed the motion of comets with John Flamsteed in terms of a fluid that revolves around the center of the cosmic system carrying the planets and comets.[7] The appearance of the 1682 comet, whose trajectory passed close to the ecliptic but in the reverse direction of planetary orbits, probably gave the final blow, in Newton's mind, to the cosmic vortex. At last, Newton realized that interplane-

---

from the Sun" are "reciprocally as the squares of the distances from the Sun," and an attempt to formulate the "lawes of motion") that are bound in Add. 3958. All these manuscripts, which date from late 1664 to late 1668, are reproduced and commented on in Herivel, *The Background to Newton's Principia* (1965). For a discussion, see Herivel, "Newton's Discovery of the Law of Centrifugal Force" (1960); Nauenberg, "Newton's Early Computational Method for Dynamics" (1994); the texts and commentary in Newton, *Unpublished Scientific Papers of Isaac Newton* (1962); Whiteside, "The Prehistory of the *Principia*" (1991); Westfall, *Never at Rest* (1980), pp. 144–54; Brackenridge, *The Key to Newton's Dynamics* (1995), pp. 42–66; and Cohen's commentary in *Principles*, pp. 11–22 and 64–70.

[6] Nauenberg, "Robert Hooke's Seminal Contributions to Orbital Dynamics" (2005).

[7] Ruffner, "Newton's Propositions on Comets" (2000).

tary space is void;[8] at any rate, there is no inert matter there. Newton never abandoned the hypothesis of the existence of a nonmaterial planetary medium completely.

Hooke's hypothesis on planetary motions was discussed at the Royal Society by astronomers interested in alternatives to Cartesian cosmology. Perhaps, as Kepler suggested, the sun was the cause of a force analogous to that active between lodestone and iron; this centripetal force would deviate the planets. Hooke surmised that the force of the sun was equivalent to terrestrial gravitation. But how could one relate this force to the observed motions of the planets? More specifically, could any mathematical implication between the three Keplerian planetary laws and a specific force law be proven? It seemed likely that this law might be inverse-square. Christopher Wren posed this problem to Halley and Hooke, asking whether either one of them, within the context of Hooke's hypothesis, could provide a mathematical theory linking the Keplerian laws to a specific force law. The winner would have been rewarded with a book worth 40 shillings.

Since finding a reply to Wren's question proved mathematically difficult, Halley took the wise, if somewhat humiliating, decision of traveling to Cambridge in August 1684 to ask Newton's advice. This episode is typical of Newton's interactions with mathematicians, theologians, chymists, and natural philosophers residing outside Cambridge. It was often the case that information could be gained from Newton by visiting him in his rooms rather than through correspondence. To his amazement, Halley found that the Lucasian Professor had an answer, or at least, so he claimed. In November 1684, Halley received a short treatise from Newton entitled "De Motu Corporum in Gyrum," in which Wren's desiderata were satisfied.[9] This is how the *Principia* began to take shape. Thanks to Halley's encouragement and insistence, a reluctant Newton was convinced to embark on a project that a couple of years later—years of hard work and scientific creativity—led to the completion of the *Principia*.

10.2 An Overview of the *Principia*

10.2.1 Definitions and Laws

The *Principia* opens with a long laudatory ode written by Halley in honor of the author. This is followed by several prefaces of philosophical content; several pages providing the basic definitions of terms like *mass* ("quantity of matter"), *momentum* ("quantity of motion"), *inherent*, *impressed*, *centripetal*, *absolute*, *accelerative*, and *motive force*; a puzzling, yet profound, Scholium on absolute time and space; the three axioms or laws of motions and their corollaries. These introductory pages have

[8] Kollerstrom, "The Path of Halley's Comet" (1999).
[9] For a commentary to *De Motu*, see De Gandt, *Force and Geometry* (1995).

attracted the attention of scholars; in particular, Newton's conceptions of absolute time and space and inherent force have been widely discussed.[10] The three laws are as follows:

> Law 1. Every body perseveres in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by forces impressed.
>
> Law 2. A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed.
>
> Law 3. To every action there is always an opposite and equal reaction; in other words, the action of two bodies upon each other are always equal and always opposite in direction.[11]

Numerous scholars have faced the question of the equivalence between Newton's second law and its modern formulation as $\vec{F} = m\vec{a}$. Note that Newton's law is formulated as a proportion, not as an equation (as in the modern case). Further, Newton's law makes no reference to time. According to some scholars, Newton's second law is best explained as the statement of a proportionality between the strength of an instantaneous impulse and a discontinuous change of momentum in the direction of the impulse. This conception of an impulsive impressed force that causes discontinuous changes of momentum might be related to Newton's endorsement of atomism, according to which impacts between hard atoms would cause instantaneous changes of velocity.

There is no doubt, however, that Newton also used the continuous formulation of the second law, as is apparent from many propositions (e.g., Proposition 6, Book 1, on central force motion (§10.2.4), where the force causes a continuous acceleration). Newton illustrated the meaning of the second axiom in a particularly interesting way in Proposition 24, Book 2:

> For the velocity that a given force can generate in a given time in a given quantity of matter is as the force and the time directly and the matter inversely. The greater the force, or the greater the time, or the less the matter, the greater the velocity that will be generated. This is manifest from the second law of motion.[12]

It is, of course, difficult to divine how Newton might have rendered such a statement using symbols. Nevertheless, his reference to both time and the quantity of matter is clearly explicit.

---

[10] See, for instance, McGuire, "Existence, Actuality and Necessity" (1978) and DiSalle, *Understanding Spacetime* (2006).

[11] *Principles*, pp. 416–417.

[12] *Principles*, p. 700. "Nam velocitas, quam data vis in data materia dato tempore generare potest, est ut vis & tempus directe, & materia inverse. Quo major est vis vel majus tempus vel minor materia, eo major generabitur velocitas. Id quod per motus legem secundam manifestum est." *Principia*, p. 432.

The most compelling evidence that Newton was close to expressing the second law in symbolic terms comes from manuscripts dating from the early 1690s.[13] In a manuscript in which Newton deals with the elementary problem of a body that moves rectilinearly in a space void of resistance and attracted by a central force (figure 10.1), we find a formulation of the second law of motion that is a step toward the expression of $F = ma$ (in one dimension) in the language of calculus:

> [Let y be the height or distance of the body from the centre [of force toward which it gravitates]. Then if the body ascends or descends straight up or down its speed will be $\dot{y}$ and gravity $\ddot{y}$. For the fluxion of the height is the body's speed and the fluxion of the speed is as the body's gravity.[14]

In the case considered by Newton, the body moves rectilinearly toward the force center, and y is the distance from the center. Newton wrote that gravity is $\ddot{y}$. This is, I claim, a prototype of $F = ma$; it is, as it were, a first significant step toward an approach to the second law in terms of fluxions. Newton, of course, along with all his contemporaries (from Johann Bernoulli to Leonhard Euler), had to write the equations of motion in the absence of a notation and calculus for vector quantities. Therefore, geometrical diagrams were used to express directionality.

Newton was able to apply the method of series and fluxions to far more complex cases. In manuscripts related to the previous one, Newton applied the use of fluxional notation and algorithm not only to the study of rectilinear motion but also to that of plane orbits traversed by a body under the action of a central force.

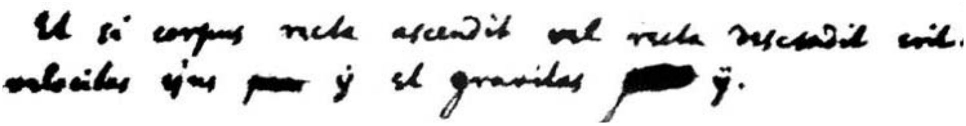


Figure 10.1
Newtonian elementary version of $F = ma$. Newton wrote the second law of motion in fluxional terms: "Et si corpus recta ascendit vel recta descendit erit velocitas ejus $\dot{y}$ et gravitas $\ddot{y}$." Source: Add. 3965.6, ff. 38r–39 (after Whiteside's intervention, now in Add. 3960.11, ff. 195–6). Reproduced by kind permission of the Syndics of Cambridge University Library.

---

[13] I have discussed these manuscripts in *Reading the Principia* (1999), pp. 108–112.

[14] Add. 3965.6, ff. 38r–39r. MP, 7, p. 129. "Prob. 2. Si corpora {gravia sunt} gravitent in centrum positione datum & lex gravitatis [pro ratione distantiarum a centro] habetur, invenire motum corporis in spatijs non resistentibus de loco dato in plagam datam data cum velocitate egressi. Exponatur tempus uniformiter fluens per longitudinem quamvis z, & sit y altitudo seu distantia corporis a centro. Cas.1. Et si corpus recta ascendit vel recta descendit erit velocitas ejus $\dot{y}$ et gravitas $\ddot{y}$. Nam altitudinis fluxio est corporis velocitas et velocitatis fluxio est ut corporis gravitas. Ideoque ex data gravitatis lege dabitur $\ddot{y}$ et inde eruendae erunt $\dot{y}$ et y. Cas. 2. Si corpus oblique moveautur i." MP, 7, p. 128.

He was able to solve several problems concerning central force motion in terms of the analytical method of fluxions. Most notably, he determined the spiral orbits traversed by a body acted upon by an inverse-cube central force. Newton communicated this last result to David Gregory in a letter dated 1694 (§10.2.7, §12.2.3, and figure 12.3). It is interesting that Newton did not pursue these researches any further and gave little emphasis to them. What is important for us, and what has turned into a sort of fixation in the historical literature, was probably deemed of secondary importance by Newton.[15]

10.2.2 Limits

The first section of Book 1 is devoted to the method of first and ultimate ratios (see §9.3 for Newton's theory of limits). Lemmas 9 and 10 are central to Newton's mathematization of force. Lemma 9 (figure 10.2) states,

> If the straight line AE and the curve ABC, both given in position, intersect each other at a given angle A, and if BD and CE are drawn as ordinates to the straight line AE at another given angle and meet the curve in B and C, and if then points B and C simultaneously approach point A, I say that the areas of the triangles ABD and ACE will ultimately be to each other as the squares of the sides.[16]

As in Lemma 7, the essential step in the proof of Lemma 9 consists in local linearization. The curve ABC can be identified in the neighborhood of A with its tangent AG, that is, the curvilinear figure ABD can be equated with the triangle

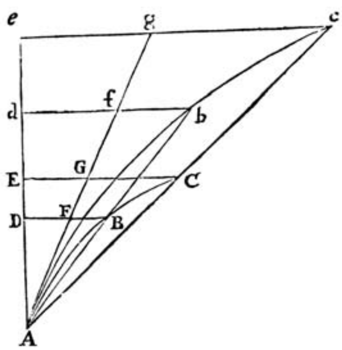


Figure 10.2

Lemma 9, Book 1. Source: Newton, *Philosophiae Naturalis Principia Mathematica* (1726), p. 33. Courtesy of the Biblioteca Angelo Mai (Bergamo).

[15] Manuscripts where Newton applies the fluxional method to central force motion can be found in MP, 6, pp. 588–93, 598–9, and are discussed in Guicciardini, *Reading the Principia* (1999), pp. 108–112.

[16] *Principles*, p. 437.

AFD. Therefore, if one takes point B close to A, the curvilinear area of ABD subtended by the curve increases, as the point D flows along Ae, very nearly as the square of "side" AD.

Lemma 9 has very important consequences for Newton's science of motion. These consequences are spelled out in Lemma 10. Let the abscissa AD in figure 10.2 represent time, and the ordinate DB velocity. Point D flows with constant velocity along the straight line AD, and the ordinate DB represents the instantaneous velocity of a body moving in a straight line (such time-velocity diagrams where known to natural philosophers familiar with the works of Galileo and Huygens). Then,

> The spaces which a body describes when urged by any finite force, whether that force is determinate and immutable or is continually increased or continually decreased, are at the very beginning of the motion in the squared ratio of the times.[17]

From Galileo's writings it was known that when the force is "determinate and immutable," the space traveled by a vertically falling body from rest is proportional to the square of time (in Newton's terminology, "the spaces described are in the squared ratio of the times"). Newton stated that this result is applicable to variable forces "at the very beginning of the motion." In the *Principia*, Newton dealt with variable forces (§10.2.4). Lemma 10 states that locally the velocity can be considered as varying linearly with time.

10.2.3 The Area Law

One of the most profound dynamical insights that Newton gained after adopting Hooke's hypothesis on planetary motions is that Kepler's area law is equivalent to central force motion. This is spelled out in Propositions 1 and 2, Section 2, Book 1. These propositions are mathematically important because, when central force motion is considered, they allowed Newton to geometrically represent time as the area swept by the radius vector, an essential step for achieving a geometrical representation of central force.

Proposition 1 is as follows:

> The areas which bodies made to move in orbits describe by radii drawn to an unmoving center of forces lie in unmoving planes and are proportional to the times.[18]

A body is fired at A with given initial velocity in the direction AB (figure 10.3). The centripetal force acting on the body must be first imagined as consisting of a series of impulses acting after equal finite intervals of time. The trajectory will then be a polygonal $ABCDEF$. The body moves, during the first interval of time, from A to B with uniform rectilinear inertial motion. If the impulse did not act

[17] *Principles*, pp. 437–8. The statement of this lemma considerably differs in the first edition.
[18] *Principles*, p. 444.

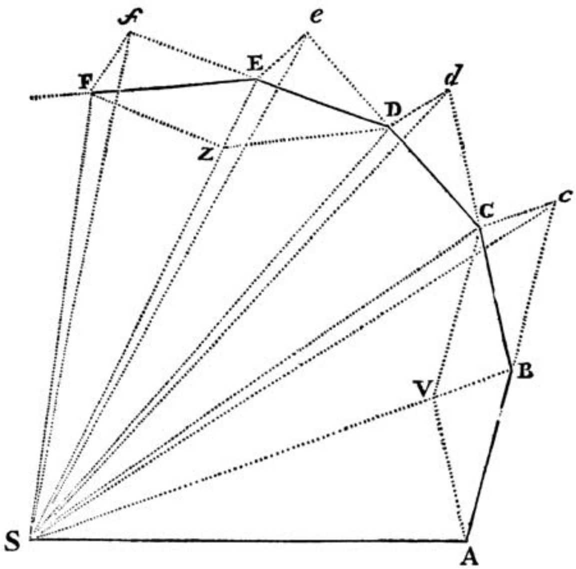


Figure 10.3
Polygonal trajectory in Proposition 1, Book 1. Source: Newton, *Philosophiae Naturalis Principia Mathematica* (1726), p. 39. Courtesy of the Biblioteca Angelo Mai (Bergamo).

at B, the body would continue its rectilinear uniform motion; it would reach c at the end of the second interval of time, so that $AB = Bc$. But because of the first impulse the body is instantaneously deflected; it reaches C at the end of the second interval of time. By applying the first two laws of motion and elementary geometry, it is possible to show that triangles SAB and SBC have the same area and lie on the same plane. Similarly, all the triangular areas SCD, SDE, SEF, etc. spanned by the radius vector in equal times are equal and coplanar. In order to prove Proposition 1, Newton took a limit. When the time interval tends to zero, the impulsive force approaches a continuous centripetal force and the trajectory approaches a smooth plane curve. The result (equal planar areas spanned in equal times by the radius vector) obtained for the polygonal trajectory generated by the impulsive force is extrapolated to the limiting smooth trajectory generated by the continuous force.

Following a similar procedure, Newton proved Proposition 2, which states the inverse of Proposition 1. In Propositions 1 and 2, Newton showed that a force is central if and only if the area law holds; the plane of orbital motion is constant and the radius vector sweeps equal areas in equal times.

Note that in his proof of Propositions 1 and 2, Newton made use of geometrical limit arguments, in line with the method of first and ultimate ratios. Indeed, it is possible to claim that Newton would not have been able to translate his demonstration of Propositions 1 and 2 into symbolic form. In certain respects, geometry was more powerful than algebra in Newton's times.

The limit argument employed in Proposition 1 did not spark any criticism in Newton's time. More recently, however, the deceptively straightforward character of this proposition has been the object of discussion.[19]

10.2.4 Central Forces

The direct problem of central forces In order to tackle central forces with geometrical methods, a geometrical representation of such forces is required. This result is not easy to acquire since the central force applied to an orbiting body changes continuously, both in strength and direction. In Proposition 6, Section 2, Book 1, such a representation is provided.

This proposition puts Hooke's hypothesis into effect. The body is accelerated *in vacuo* by a central force, and its motion, as Hooke had suggested, is decomposed into an inertial motion along the tangent and an accelerated motion toward the force center.

A body accelerated by a centripetal force directed toward S (the center of force) describes a trajectory like the one schematically shown in figure 10.4. PQ is the arc traversed in a finite interval of time. The point Q is fluid in its position on the orbit, and one has to consider the limiting situation when points Q and P come together. Line ZPR is the tangent to the orbit at P. QR tends to become parallel to SP as Q approaches P. QT is normal to SP. From Lemma 10, "at the very

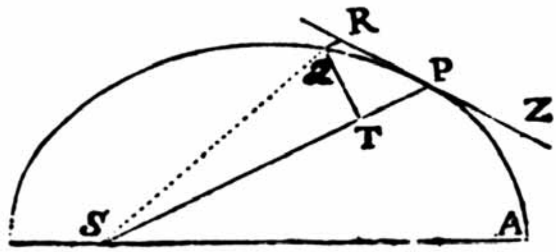


Figure 10.4
Central force motion in Proposition 6, Book 1. Source: Newton, *Philosophiae Naturalis Principia Mathematica* (1687), p. 44. Courtesy of the Biblioteca Angelo Mai (Bergamo).

[19] The essential aspect to bear in mind is that Newton assumed that a continuous trajectory is given and used the polygonal trajectories as approximations of a *given* smooth trajectory. His limit argument was not meant to generate an unknown continuous trajectory. Nauenberg, "Kepler's Area Law in the *Principia*" (2003); Pourciau, "Newton's Argument for Proposition 1 of the *Principia*" (2003).

beginning of the motion" the force can be considered constant. In the case represented in figure 10.4, this implies that *as Q approaches P, the displacement QR is proportional to force times the square of time.* Indeed, in the limiting situation, QR can be considered as a small Galilean fall caused by a constant force.

Newton could now obtain the required geometrical representation of force. Since Kepler's area law holds (see Proposition 1 in §10.2.3), the area of SPQ is proportional to time. Further, SPQ can be considered a triangle, since the limit of the ratio between the vanishing chord PQ and arc $\widehat{PQ}$ is 1 (see Lemma 7 in §9.3). The area of triangle SPQ is $(SP \cdot QT)/2$. Therefore, the geometrical measure of force is

$$F \propto \frac{QR}{(SP \cdot QT)^2}, \tag{10.1}$$

where the ratio has to be evaluated in the limiting situation when points P and Q come together and $\propto$ is used to mean "is proportional to."

Proposition 6 is a good example of the application of the method of first and ultimate ratios. The limit to which the ratio $QR/(SP \cdot QT)^2$ tends is to be evaluated by purely geometrical means. Note that SP remains constant as Q tends to P; therefore one has to consider the limit of the ratio QR/QT^2.

When the trajectory O is an equiangular spiral (in polar coordinates $\ln r = a\theta$) and S is placed at the center (Proposition 9, Section 2), as Q tends to P,

$$QR/QT^2 \propto 1/SP, \tag{10.2}$$

and thus the force varies inversely with the cube of distance.

When the trajectory O is an ellipse and the center of force lies at its center (Proposition 10, Section 2), $QR/QT^2 \propto SP^3$, that is, the force varies directly with distance.

Note that forces varying with the inverse of the cube of the distance later found an application, in Book 3, in the study of tidal forces, and forces that vary directly with distance occurred in the study of elastic vibrations.

In Section 3, Newton considered Keplerian orbits. In Proposition 11, Newton proved that if the body describes a trajectory O, O is an ellipse and the force is directed toward a focus S, then the force varies inversely with the square of the distance. In Propositions 12 and 13, Newton showed that the force is also inverse-square if O is a hyperbola or parabola. To conclude: when the orbit is a conic section and S is placed at one focus,

$$QR/QT^2 \propto 1/L, \tag{10.3}$$

where L is a constant (the *latus rectum*), and the ratio QR/QT^2 is evaluated, as always, as the first or last ratio with the points Q and P coming together. Therefore, the strength of a force that accelerates a body obeying the first two

Keplerian laws varies inversely with the square of distance. This is often considered to be the birth of gravitation theory, even though, as experts know, Propositions 11–13, Book 1, played a limited role in Newton's deduction of universal gravitation from astronomical phenomena.[20]

The inverse problem of inverse-square central forces In Corollary 1 to Propositions 11–13, Section 2, Newton stated that if the central force is inverse-square, then the orbits of a body accelerated by such a force are conic sections such that a focus coincides with the force center. Corollary 1 is as follows:

> From the last three propositions [Propositions 11–13] it follows that if any body P departs from the place P along any straight line PR with any velocity whatever and is at the same time acted upon by a centripetal force that is inversely proportional to the square of the distance of places from the center, this body will move in some one of the conics having a focus in the center of forces; and conversely.[21]

This Corollary is an example of what was called an inverse problem of central forces. The central force F (force law and force center S) is known. What is required is the orbit ("the orbit" is a singular, assuming uniqueness) corresponding to any initial position and velocity of a body with a given mass acted upon by such force.

Quite understandably, this terse statement was subject to criticism. Newton himself, while preparing the second edition of the *Principia*, emended Corollary 1 by adding the sketch of what he considered to be a valid proof. In a letter, dated October 1709, he instructed Roger Cotes to complete Corollary 1 with the following lines:

> For if the focus and the point of contact and the position of the tangent are given, a conic can be described that will have a given curvature at that point. But the curvature is given from the given centripetal force [and velocity of the body]; and two different orbits touching each other cannot be described with the same centripetal force [and the same velocity].[22]

What Newton cryptically stated in Corollary 1, Book 1, can perhaps be elucidated as follows:[23]

1. Given a body whose mass is m fired from a given point P and with a given velocity $\vec{v}$ (assumed as not directed toward the force center) in a central inverse-square

---

[20] For Newton's deduction of universal gravitation, see Harper, "Newton's Argument for Universal Gravitation" (2002) and Smith, "The Methodology of the *Principia*" (2002). For more elementary analyses of Newton's mathematization of central force motion in the first three sections of Book 1 of the *Principia*, see Brackenridge, *The Key to Newton's Dynamics* (1995) and Densmore, *Newton's Principia, the Central Argument* (1995).

[21] *Principles*, p. 467.

[22] *Principles*, p. 467. The text in square brackets was added in the third (1726) edition.

[23] Pourciau, "On Newton's Proof That Inverse-Square Orbits Must Be Conics" (1991) and "Newton's Solution of the One-Body Problem" (1992).

force field (F force strength, S center of force), Newton sought a geometrical technique for the construction of the orbit traversed by the body.

2. Initial position and velocity determine a point P belonging to the trajectory and the trajectory's tangent at P.
3. Since the strength and direction of the force at P are given, also the curvature at P is uniquely determined from initial conditions ($F_N = mv^2/\rho$, F_N normal component of force at P, ρ radius of curvature at P).
4. In Proposition 17, Book 1, Newton showed how to construct a unique conic, with known tangent and curvature at P (determined in steps 2 and 3) and a focus located at the force center S.
5. Initial conditions also determine the areal velocity, which, since the force is central, must be a constant of motion (Propositions 1 and 2, Book 1).[24]
6. Because of Propositions 11–13, Book 1, the Keplerian orbit determined in step 5 satisfies the equations of motions.
7. Since for every initial condition—position and velocity—a Keplerian orbit that satisfies the equations of motions can be constructed, one derives (assuming that for every initial condition only one orbit is possible) that Keplerian orbits are necessary in a central inverse-square force field.

Newton expressed the assumption of uniqueness (implicit in point 7) in the final lines of the revised corollary ("[T]wo different orbits touching each other cannot be described with the same centripetal force and the same velocity"). It seems that for Newton it was intuitively legitimate to state that, given initial position and velocity, only one orbit is possible. Further, he might have deduced uniqueness from Proposition 41, Book 1, where the inverse problem is reduced to quadratures (§10.2.7).[25]

Corollary 1 never ceased to be an object of criticisms and debates. But Newton outlined a more general approach to the inverse problem of central forces (§10.2.7), an approach that inspired research in the mathematization of central force motion in terms of differential equations.

10.2.5 Projective Geometry

In Sections 4 and 5, Book 1, several propositions are devoted to the geometry of conic sections. It is here that Newton presented his results on projective geometry.

[24] In Section 6, Book 1, Newton showed how a motion satisfying Kepler's area law along a given conic can be approximated (§10.2.6).

[25] For a recent discussion of the uniqueness assumption in Corollary 1, see Pourciau, "Proposition II (Book I) of Newton's *Principia*" (2009), p. 164–166. On page 166, Pourciau expresses the uniqueness assumption as follows: "[T]wo different motions, urged by the same centripetal force throughout, cannot pass through the same given point P at the same time and with the same speed and direction."

Lemma 19 is a solution of the Pappus problem attained not by a "computation [as Descartes had done] but a geometrical synthesis, such as the ancients required" (§5.3).[26] Also notable are Lemma 21, on the organic construction of conics (§5.4), and Lemma 22, on the projective "transmutations" of figures (§6.4.2).

10.2.6 Algebraic Nonintegrability of Ovals

Section 6, Book 1, of the *Principia* is devoted to the solution of the so-called Kepler problem. The problem consists in finding the area of a focal sector of the ellipse and is equivalent to the solution for x of the equation $x - e\sin x = z$ (e and z given). Johannes Kepler found that planets move in ellipses having the sun placed at one focus. He also discovered that each planet moves in such a way that the radius vector joining it to the sun sweeps equal areas in equal times. When the elliptic orbit is known, the position of the planet in function of time can thus be found by calculating the area of the focal sector. In Lemma 28, Section 6, Newton demonstrated that this problem cannot be resolved in finite algebraic terms (§13.3). In Proposition 30, Section 6, he showed that the determination of the position of a body orbiting in a parabolic trajectory (such that the area law is valid for the focus) is instead algebraic (see chapter 11). But to deal with the Kepler problem for elliptic trajectories, polynomial equations are not enough. In Section 6, Newton therefore illustrated how the roots of the Kepler's equation can be determined via his method of successive approximations, which is to say, via infinite series (see §7.5, figure 7.10).[27] Infinite series occur throughout the *Principia* (especially in the final Scholium to Section 13, Book 1; Proposition 45, Book 1; and Proposition 10, Book 2), and Newton might well have had this in mind when he stated that his work was based on new analysis. He also employed quadrature techniques, another key element of his new analysis.

10.2.7 The General Inverse Problem of Central Forces

In Sections 7 and 8, Book 1, Newton faces the general inverse problem of central forces: that is, the problem of determining the orbit, given initial position and velocity, of a body accelerated by a known central force. He first dealt with "rectilinear ascent and descent" and then with curvilinear motion.

The inverse problem for inverse-square central forces had already been dealt with by Newton in Corollary 1 to Propositions 11–13 and in Proposition 17, Book 1 (§10.2.4). But in Proposition 41, Book 1, a *general* solution to the inverse problem is provided:

[26] Corollary 2, Lemma 19, Section 5, Book 1, in Newton, *Principles*, p. 485.

[27] See Adams, "On Newton's Solution of Kepler's Problem" (1882); Kollerstrom, "Thomas Simpson and Newton's Method of Approximation" (1992).

> Supposing a centripetal force of any kind and granting the quadratures of curvilinear figures, it is required to find the trajectories in which bodies will move and also the times of their motions in the trajectories so found.[28]

This proposition is based on the assumption that a method for the "quadrature of curvilinear figures" is given. In his youth, Newton had developed several quadrature techniques (integration) as part of the inverse method of fluxions. However, in the *Principia*, Newton chose not to make his mathematical discoveries in this field entirely explicit.

When in the *Principia* he reduced a problem to a difficult quadrature, he was following the practice of giving the composition without the resolution; the synthesis without the analysis. He simply showed that the solution depends upon the quadrature of a curve (that is, upon the determination of the area bounded by a curve) and gave no hint on how to perform the required quadrature. Other examples of these mysterious reductions to quadratures can be found in Newton's treatment of the attraction of extended bodies (Sections 12 and 13, Book 1), of the solid of least resistance (Scholium to Proposition 35, Book 2), and in most of Book 3, particularly in the study of the inequalities of the moon's motion (Propositions 26–35, Book 3). These sections of the *Principia* puzzled Newton's readers; while they were told that a result depends upon the quadrature of a certain curve, they were not given the method by which this quadrature could be achieved.[29] Chapter 12 examines Newton's policy on the publication of analytical quadratures.

As far as Proposition 41 is concerned, the following points should be emphasized:

1. Its statement and demonstration are geometrical, but can easily be translated into calculus by substituting symbols for infinitesimal geometrical moments. One can choose to employ either the Newtonian notation ($\dot{x}o$) or the Leibnizian one (dx).[30]
2. The final result, as printed in the *Principia*, is easily translatable into a couple of fluxional (or differential) equations.
3. Newton was aware that a translation into the analytical method of fluxions was feasible. Overwhelming evidence is there to prove it.

    a. In the statement of the proposition Newton wrote that the demonstration of Proposition 41 depends upon the quadrature of curvilinear figures, a reference to his inverse method of fluxions. A number of propositions in the *Principia* actually begin by stating that the demonstration presupposes an

[28] *Principles*, p. 529.
[29] Guicciardini, *Reading the Principia* (1999).
[30] Such a translation is provided in Cohen, "A Guide to Newton's *Principia*" (1999), pp. 334–45.

available method for the squaring of curves (*concessis curvilinearum figurarum quadraturis*).

b. In Corollary 3, Newton applied the general result of Proposition 41 to the case of an inverse-cube force. The question was, Which orbits are described by a body accelerated by an inverse-cube force? In Corollary 3, Newton printed only the solution in the form of a geometrical construction. Following his usual publication policy, he did not reveal the new analysis; rather, he gave the construction (synthesis). Newton thus constructed some trajectories that answered the problem. He could have obtained this result only by applying his catalogues of curves (in Leibnizian terms, integral tables) (§12.2). The general result achieved in Proposition 41, when applied to an inverse-cube force, leads to the quadrature of a curve included in the catalogues of *De Methodis* (1671). In Corollary 3, Newton did not perform this quadrature explicitly but simply stated the result. He then added,

   > All this follows from the foregoing proposition [41], by means of the quadrature of a certain curve, the finding of which, as being easy enough, I omit for the sake of brevity.[31]

   This is one of the most famous instances of those gaps in the demonstrative structure of the *Principia* that aroused the interest and frustration of its readers.[32]

c. When David Gregory, during a visit paid to Newton in May 1694, asked about the mysterious method applied in the construction of Corollary 3, the Lucasian Professor answered by translating the basic result of Proposition 41 into two fluxional equations (see figure 12.3).[33] He applied these equations to the case of an inverse-cube force and obtained the construction printed in the *Principia* as Corollary 3. As Gregory remarked in a memorandum of this visit,

   > The second treatise [*De Quadratura*] will contain his [Newton's] Method of Quadratures ... on these [quadratures] depend certain more abstruse parts in his philosophy as hitherto published, such as Corollary 3, Proposition 41 and Corollary 2, Proposition 91.[34]

   For an analysis of these corollaries see chapter 12.

---

[31] *Principles*, p. 532.

[32] On Corollary 3, Proposition 41, Book 1, see Whiteside, "The Mathematical Principles Underlying Newton's *Principia Mathematica*" (1970); Erlichson, "The Visualization of Quadratures in the Mystery of Corollary 3 to Proposition 41 of Newton's *Principia*" (1994); and Brackenridge, "Newton's Easy Quadratures" (2003).

[33] *Correspondence*, 6, pp. 435–7.

[34] Memorandum, July 1694 (Edinburgh University Library, MS Gregory C42). Translation from Latin in *Correspondence*, 3, p. 386. For a discussion of Gregory's comments to the *Principia*, see Guicciardini, *Reading the Principia* (1999), pp. 179–84.

10.2.8 Advanced Problems

In order to approach universal gravitation in mathematical terms, Newton had to advance into unknown territory. Until Section 8, Book 1, he dealt with a body moving in a central force field in the absence of resisting media. Newton knew that this mathematical model could be applied only approximately to the planetary system. In practice, if one takes a system composed of two bodies, 1 and 2, sufficiently far both from other disturbing bodies and one another, where 1 has a much greater mass than 2, it is possible to approximate 1 as an immovable center of force and 2 as a point mass. This simplified model is also employed in Section 9, Book 1, devoted to the motion of the line of apsides occurring when the body is accelerated by a central force that varies as $1/r^{2+\alpha}$, for a small α. It is only in Section 11 that Newton considered the motion of two or more bodies that mutually attract each other; and only in Sections 12 and 13 that he paid attention to the shape of the bodies and to the gravitational force exerted by such extended bodies.

These more advanced sections of Book 1 contain a wealth of results, especially on perturbation theory.[35]

Book 2 is devoted to the motion of bodies in resisting media. It is rich in mathematical results, most notably in the Scholium to Proposition 35 (= 34, from 2d ed.) Newton was a pioneer in variational methods when he tackled the problem of the solid of least resistance. The concluding Section 9 leads to what Newton conceived of as a refutation of the vortex theory of planetary motions. Book 2 contains many pages devoted to experimental results on resisted motion. The mathematical parts of Book 2 are extremely problematic. Compared with the mathematical methods of the first Book, those of the second were considered, ever since Newton's times, less satisfactory.

In Book 3, Newton applied the mathematical results achieved in his first Book to astronomy. In a sequence of opening propositions he was able to infer from astronomical data that planetary motions are caused by a gravitational force. This force acts instantaneously, in a void, and attracts two given point masses with a strength proportional to the product of the masses and inversely proportional to the square of their distance.

In the remaining part of Book 3, Newton, assuming the existence of such a force between any two masses in the whole universe, was able to provide quantitative estimates of diverse phenomena such as the motion of tides, the shape of the earth, some of the inequalities of the moon's motion, the precession of equinoxes, and the trajectories of comets. In Lemma 5, in dealing with cometary paths, Newton presented a method of interpolation that inspired researches by mathematicians such

[35] Nauenberg, "Newton's Perturbation Methods for the Three-Body Problem" (2001) and Wilson, "Newton on the Moon's Variation and Apsidal Motion" (2001).

as James Stirling, Friedrich Wilhelm Bessel, and Carl Friedrich Gauss (see §8.6, figure 8.20).[36]

Perhaps the most controversial aspect of Book 3 is Newton's theory on the moon, especially as regards the motion of the moon's apse. In 1702, Newton published a booklet devoted to the "theory" of the moon, a very important problem for the determination of longitude.[37]

10.3 Did Newton Use the Calculus in the *Principia*?

10.3.1 Addressing a Disputed Question

Before considering Newton's use of common analysis and new analysis in the *Principia* (see chapters 11 and 12), I address a disputed question concerning the use of the calculus in Newton's *magnum opus*. I clarify some general points before examining the mathematical details.

It is appropriate to begin with a remark by Whiteside, the greatest historian of Newton's mathematics:

> How often am I still asked: "*Did* Newton use the calculus to obtain the theorems in his *Principia*?" How, without seeming to patronize, do you lay the groundwork on which you can reply that the question is ill-formed and therefore meaningless.[38]

Let me clarify why, with Whiteside, I deem the question to be ill-formed.

1. As should be clear from the overview of the *Principia* provided in section 10.2, Newton employed a variety of mathematical methods: projective geometry (Sections 4 and 5, Book 1); geometrical limits of vanishing magnitudes (e.g., in Propositions 1, 6, and 11–13, Book 1); infinite series expansions (e.g., in Proposition 45, Book 1, and Proposition 10, Book 2); algebraic equations (e.g., in Proposition 30, Book 1); quadratures of curvilinear figures (e.g., Corollary 3 to Proposition 41 and Corollary 2 to Proposition 91, Book 1); calculation of the radii of curvature (e.g., Proposition 28, Book 3).[39] Contrary to what one might

[36] Newton's interpolation formulas were published under the title of *Methodus Differentialis* in Newton, *Analysis per Quantitatum* (1711), pp. 93–101. For a treatment of *Methodus Differentialis*, see Fraser, *Newton's Interpolation Formulas* (1927), and Whiteside's commentary in MP, 4, pp. 36–51.

[37] Reprinted with commentary in Cohen, *Isaac Newton's Theory of the Moon's Motion* (1975). See also Kollerstrom, *Newton's Forgotten Lunar Theory* (2000).

[38] Whiteside, "The Prehistory of the *Principia*" (1991), p. 11.

[39] I discuss Propositions 30, 41, and 91, Book 1, in chapters 11 and 12. For Proposition 45, Book 1, see Valluri, Wilson, and Harper, "Newton's Apsidal Precession Theorem and Eccentric Orbits" (1997); for Proposition 28, Book 3, see Nauenberg, "Newton's Perturbation Methods for the Three-Body Problem" (2001).

be led to believe by inspecting the *Principia* superficially, Newton's *magnum opus* does not rest upon one single overarching mathematical method.

2. So bluntly posing the question of Newton's use of the calculus in the *Principia* is rather problematic, since it is far from clear what *calculus* meant for his contemporaries. In order to formulate a question that is historically meaningful, one must accept the fact that Newton's method of series and fluxions does not coincide with Leibniz's differential and integral calculus; if one searches for the latter in Newton's work, the search is bound to fail. Newton and his supporters defended an approach to mathematics that needs to be understood on its own terms.[40]
3. Newton did not present the *Principia* as the unfolding of a deductive theory whereby theorems are deduced from axioms. He rather divided his propositions into theorems and problems. To tackle these problems, Newton made use of analytical tools of resolution; consequently he employed not the term *calculus* but rather *method*. One therefore should not expect to find "calculus" applied to the demonstration of Newton's theorems in the *Principia* but rather his method of series and fluxions applied to the resolution and composition of some (but not all) of the problems.
4. In many demonstrations of the *Principia*, Newton had recourse to a wide variety of geometrical methods (which he either found in the works of Apollonius and Euclid or invented himself by developing results in projective geometry and his synthetic method of fluxions). Particularly in the first sections of the *Principia*, for instance, Proposition 1 (§10.2.3) and Proposition 6 (§10.2.4), Newton inventively and rather effectively proceeded "without calculation." The necessity of employing symbolical techniques surfaced in the more advanced sections of Book 1, and especially in Book 3, where Newton tackled very difficult problems with the purpose of obtaining numerical predictions to test his theory of gravitation.

When the debate surrounding the priority of Newton over Leibniz in the invention of the calculus had taken off, the use of differential and integral calculus in the *Principia* became a hot issue. According to the Leibnizians, and particularly Johann Bernoulli, the *Principia* offered positive proof of the fact that Newton did not know about the calculus in 1687, since any use of it, in Bernoulli's opinion, was conspicuously absent in the work. Newton's reply was to state that he had indeed used the analytical method of series and fluxions (the new analysis) throughout the work but had not rendered it explicit for two reasons:

1. In his replies to Leibniz and Johann Bernoulli, Newton made it clear that in the *Principia* he was addressing himself to readers who were not prepared to engage with such a novel mathematical language. To present a new science of motion

[40] See Guicciardini, *Reading the Principia* (1999).

and a new cosmology in a mathematical language that was almost unknown would have been prohibitive even for the most numerate of his contemporaries.

2. Newton affirmed that in writing the *Principia* he had chosen, following the practice of the ancients, to demonstrate synthetically its propositions in order to convey mathematical certainty to natural philosophy. He also claimed that most of the propositions had been first found by help of the new analysis, a statement that can only be defined as excessive.

10.3.2 The Readers of the *Principia*

It is certainly true that in 1687 adopting geometry was the most natural choice in writing a work devoted to the mathematization of natural philosophy. The works of Galileo and Huygens, Newton's greatest predecessors in the tradition of physico-mathematical science, were written in geometrical terms. However, in a period of tumultuous transition like the seventeenth century, geometry was a term as ambiguous as calculus.

The geometry of the *Principia* might look familiar at a superficial inspection. Yet those who tried to read the *Principia* in depth soon discovered that Newton's geometry was far removed from the beaten track. Figures are generated by continuous motion, and limits of ratios and sums of vanishing magnitudes occur throughout the work—characteristics that are totally innovative in comparison to ancient geometry and that depend upon Newton's desire, manifested in the "Geometria Curvilinea," to extend the canon of Euclid's *Elements*.

Further, the geometrical demonstrative structure of the *Principia* could not conceal its gaps: the presence of statements lacking any demonstration. In order to fill these gaps, the use of algebra, series, higher-order derivatives, and integration is necessary. Very few details regarding these algorithmic techniques were provided in the printed text.

Both Newton's British acolytes and his fierce continental critics were interested in the missing algorithmic steps. Newton might have easily added some details on the use of algebraic equations; these were certainly techniques well known to mathematically trained readers in 1687 (see chapter 11 for an example). Newton might also have given more indications on the use of higher algorithmic techniques, such as series and quadratures, for instance, in an Appendix. He actually considered this option in the 1690s when he began considering a revised second edition. But even in its second (1713) and third (1726) editions, the *Principia* was still written in a geometrical language, a geometrical veil that kept what seemed mathematically more interesting out of sight.

Continentals such as Leibniz, the Bermoullis, and Pierre Varignon took a completely different direction, promoting the use of calculus in "dynamics." But even in Newton's circle mathematicians such as David Gregory, Roger Cotes, Brook Taylor, James Stirling, and Abraham De Moivre were consulting their master about the use of fluxions and series in the study of motion. They shared ideas within their close circle and often competed with the Continentals in the use of fluxional/differential equations in the study of motion in nonresisting and resisting media.

The competences of mathematicians changed rapidly during Newton's mature years. The *Principia* might have been written with the mathematical background of its 1687 readers in mind, but, under Newton's disconcerted gaze, its mathematical style soon became obsolete. As Newton wrote in the late 1710s,

> To the mathematicians of the present century, however, versed almost wholly in algebra as they are, this [the *Principia*'s] synthetic style of writing is less pleasing, whether because it may seem too prolix and too akin to the method of the ancients, or because it is less revealing of the manner of discovery. And certainly I could have written analytically what I had found out analytically with less effort than it took me to compose it. I was writing for Philosophers steeped in the elements of geometry, and putting down geometrically demonstrated bases for physical science. And the geometrical findings which did not regard astronomy and physics I either completely passed by or merely touched lightly upon.[41]

While the philosophers in 1687 were steeped in geometry, the younger mathematicians, formed at Bernoulli's school in Paris and Basel, who began their studies in higher mathematics reading L'Hospital's *Analyse des Infiniment Petits* (1696), found the *Principia* obscure.

10.3.3 A Method for Making Things Certain

I now turn to the second part of Newton's reply to the critics of the *Principia*, who used his *magnum opus* to demonstrate his lack of knowledge of calculus. During his dispute with Leibniz, speaking of himself in the third person, Newton anonymously stated,

> By the help of this new Analysis Mr Newton found out most of the Propositions in his *Principia Philosophiae*: but because the Ancients for making things certain admitted nothing into Geometry before it was demonstrated synthetically, he demonstrated the Propositions synthetically, that the Systeme of the Heavens might be founded upon good Geometry. And this makes it now difficult for unskillful men to see the Analysis by which those Propositions were found out.[42]

---

[41] MP, 8, p. 451. "Mathematicis autem hujus saeculi, qui fere toti versantur in Algebra, genus hocce syntheticum scribendi minus placet, seu quod nimis prolixum videatur & methodo veterum nimis affine, seu quod rationem inveniendi minus patefaciat. Et certe minori cum labore potuissem scribere Analytice quam ea componere quae Analytice inveneram: sed propositum non erat Analysin docere. Scribebam ad Philosophos Elementis Geometriae imbutos & Philosophiae naturalis fundamenta Geometrice demonstrata ponebam. Et inventa Geometrica quae ad Astronomiam et Philosophiam non spectabant, vel penitus praeteribam, vel leviter tantum attingebam." MP, 8, p. 450. I have slightly altered Whiteside's translation.

[42] Newton, "Account" (1715), which Newton anonymously published in the *Philosophical Transactions*. Reprinted in Hall, *Philosophers at War* (1980), p. 296. See also MP, 8, pp. 598–9.

It would be naive, of course, to take such statements literally. Newton was writing in the muddied context of the priority dispute; he was far from objective. Yet these and similar pronouncements should not be dismissed by the historian, and for two very good reasons.

First, they are precious in revealing some of the values that Newton wished to defend in his mature years, most notably, the notion that the "good geometry," upon whose solid foundations he had developed his natural philosophy, must be consistent with the synthesis of the ancients.

Second, these pronouncements show that Newton was claiming to be following a canon of problem solving whereby the analysis that allows one to answer a question should be neglected in the synthetic, demonstrative composition. As Newton stated,

> Solution is, however, the opposite of resolution in that it may not be had till all trace of resolution be removed from start to finish by means of a full and perfect composition. For example, if a question be answered by the construction of some equation, that question is resolved by the discovery of the equation and composed by its construction, but it is not solved before the construction's enunciation and its complete demonstration is, with the equation now neglected, composed.[43]

This canon was at work in *Arithmetica Universalis*, where the common (finite) Cartesian analysis was to be neglected in the construction of equations (see chapter 4). It was at work in *Enumeratio Linearum Tertii Ordinis*, where the algebraic and fluxional analysis that lies behind many of Newton's statements was only obliquely revealed to the reader (see chapter 6). This canon was also at work in *De Methodis*, where the analytical solution of quadrature problems was followed by demonstrations of the constructions in which no algebraic calculation occurs, since only such constructions are worthy of public utterance (§9.1).[44] Newton claimed that all synthetic constructions of quadratures in the *Principia* (see two examples in chapter 12) were first discovered by analytical means, namely, by highly algorithmic quadrature techniques (§8.4). A close scrutiny of the *Principia* shows this claim to be excessive.[45] In some instances, it is nevertheless possible to find internal and external evidence that the analytical method of quadratures was at work in the *Principia*.

Parts V and VI expand on the values that Newton defended in his mature published and unpublished mathematical works. Chapters 11 and 12 present some

---

[43] MP, 7, p. 307. "solutionem vero ita contrariam esse resolutioni ut ea non prius habeatur quam resolutio omnis a principio ad finem per compositionem plenam et perfectam excludatur. Verbi gratia si quaestioni per constructionem aequationis alicujus respondeatur, quaestio illa resolvitur per inventionem aequationis, componitur per constructionem ejusdem, sed non prius solvitur quam constructionis enunciatio ac demonstratio tota componitur, aequatione neglecta." MP, 7, p. 306.
[44] MP, 3, p. 279.
[45] See Guicciardini, *Reading the Principia*.

examples of hidden analysis in the *Principia*. Note, however, that use of symbolical analysis that can be identified in some propositions of the *Principia* is the exception, not the rule (as Newton claimed during the priority dispute). Newton made use of algebraic and fluxional analyses in certain demonstrative passages of the *Principia*, yet his work was not systematically written in symbolical terms, as was done by Continental mathematicians from Varignon to Euler and beyond.[46] By the middle of the eighteenth century mathematicians such as Euler and d'Alembert conceived analytical mechanics as a discipline entirely written in calculus terms. Their starting point was the differential equations of motion, which were subsequently solved for particular boundary and initial conditions. For instance, in tackling the three-body problem, as Wilson made clear, they "started from differential equations that stated exactly the conditions of the problem," and in their mathematical practice "reference to the differential equations . . . controls the successive approximations."[47] By contrast, in Newton's *Principia* methods equivalent to Leibnizian integrations do occur but only sporadically in a demonstrative context that is ultimately grounded on physical and geometrical insights.

[46] Blay, *La Naissance de la Mécanique Analytique* (1992).
[47] Wilson, "Newton on the Moon's Variation and Apsidal Motion" (2001), p. 153.

11 Hidden Common Analysis

> As a matter of fact, even though Newton's *Principia* in many places offers examples of the ancient procedure, in general the calculus surfaces behind the concealment with which Newton keeps it hidden from sight. This is a drawback which is frequent in those books delivered as writings faithful to the ancient method, but which are, in fact, just disguised algebra.
>
> —Jean E. Montucla, 1802

11.1 Proposition 30, Book 1, of the *Principia*

This chapter considers how Newton employed algebraic equations in the *Principia.* He did employ algebra, which for several purposes proved to be a useful analytical tool. But he did not print the Cartesian (or common) analysis but rather the geometrical synthesis, that is, the *compositio* but not the *resolutio.* As explained in chapter 3, the Cartesian canon of problem solving required that, after having reduced a problem to an algebraic equation, one had to produce a geometrical construction followed by a geometrical demonstration that the construction solved the problem.

The use of algebraic equations was well known to even modestly competent readers in 1687 Britain. Textbooks on the use of algebraic equations, such as Oughtred's *Clavis* (1631), which ran numerous editions, Johann Heinrich Rahn's *An Introduction to Algebra* (1668), and John Kersey's *The Elements of That Mathematical Art Commonly Called Algebra* (1673–1674), were easily available. Newton's choice to conceal what was called, because of its well-established status, common analysis did not depend upon worries about the competences of his readers. It was a choice dictated by a well-defined publication strategy.

Epigraph from Montucla, *Histoire des Mathématiques* (1799–1802), 3, p. 6."En effet, quoique ses principes [*Principia*] nous offrent en bien des endroits des exemples de ce tour ancien; en général le calcul y perce à travers le déguisement dont Newton l'a couvert, espèce de défaut, commun à bien des livres donnés pour écrits suivant la méthode ancienne, et qui ne sont que de l'algèbre déguisée." When Montucla died, pages 1–336 of volume 3 of his new edition of the *Histoire* had already been proofread and printed. The rest was revised by J. de Lalande (Lalande availed himself of the help of several scholars; most notably S. F. Lacroix revised pages 342–352 on integration of partial differential equations. See footnotes on pages 336, 342, 344, 349 in volume 3). Since volumes 3 and 4 of the *Histoire* are a cooperative effort, it is improper to attribute them to a single author. We do not know how heavily Montucla's text was changed, especially after page 336. It is fair, I surmise, to attribute to Montucla quotations from pages 1–336 of volume 3.

An example of hidden common analysis occurs in Proposition 30, Book 1. This proposition deals with a simple problem, one that can be solved in finite algebraic terms. A body moves along a given parabola AP, where A is the vertex and S the focus (figure 11.1). The variable position of the body is indicated by P. The radius vector SP sweeps equal areas in equal times. Also known are the time when the body is at A and the initial velocity at A. The problem is how to find the position P of the body as a function of time. Newton solved this problem by a geometrical construction that, according to Chandrasekhar, "passes understanding."[1]

Newton's prescriptions were quite simple. One must draw a straight line passing through G (the midpoint between the vertex A and the focus S) and orthogonal to the parabola's axis. A point H flows with constant velocity along the straight line so that its velocity is three-eighths the initial velocity of the body at the vertex A. In order to determine the position of H as a function of time, one needs some initial condition. Newton prescribed that the position of the fluent point H is G, when the body is at the vertex A. Next, one is required to draw, at any time, a circle with center H and radius $HA = HS$. The circle will cut the parabola in the point P which determines, as required, the position of the body along the parabolic trajectory.

After providing this geometrical construction, Newton demonstrated that it was exactly what was required. What Newton did not do is explain how he found that construction. In other words, the reader of the *Principia* found a successful geometrical construction but no hint as to how such construction was achieved by

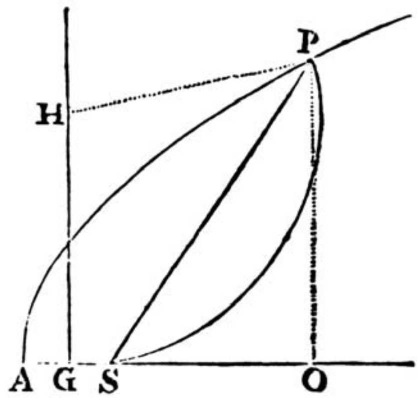


Figure 11.1
Newton's diagram for Proposition 30. Source: Newton, *Philosophiae Naturalis Principia Mathematica* (1726), p. 105. Courtesy of the Biblioteca Angelo Mai (Bergamo).

[1] Chandrasekhar, *Newton's Principia for the Common Reader* (1995), p. 131.

its author. As is typical with synthetic constructions, it "passes understanding." However, a reader who has some knowledge of seventeenth-century mathematics will not spend too much time in recognizing a technique typical of Descartes' *Géométrie* in Newton's construction. It goes without saying that Newton knew this technique, since in his youth he had read the *Géométrie* with avidity. Further, in his *Lucasian Lectures on Algebra*, deposited in the University Library in 1684 and published in 1707 as *Arithmetica Universalis*, Newton devoted a great deal of space to the algebraic resolution of problems and to the construction of equations (§4.3, §4.4).[2]

11.2 Newton's Synthetic Construction for Proposition 30, Book 1

It is worth quoting Newton at length in order to allow the reader to appreciate his synthetic procedure:

> Proposition 30. Problem 22.
>
> *If a body moves in a given parabolic trajectory, to find its position at an assigned time.*
>
> Let S be the focus and A the principal vertex of the parabola [see figure 11.1], and let $4AS \times M$ be equal to the parabolic area APS to be cut off, which either was described by the radius SP after the body's departure from the vertex or is to be described by that radius before the body's arrival at the vertex. The quantity of that area to be cut off can be found from the time, which is proportional to it. Bisect AS in G, and erect the perpendicular GH equal to $3M$, and a circle described with center H and radius HS will cut the parabola in the required place P. [Author's note: this ends the construction of the problem. What follows until Q.E.D. is the demonstration of the construction.]
>
> For, when the perpendicular PO has been dropped to the axis and PH has been drawn, then $AG^2 + GH^2$ $(= HP^2 = (AO - AG)^2 + (PO - GH)^2) = AO^2 + PO^2 - 2GA \times AO - 2GH \times PO + AG^2 + GH^2$. Hence $2GH \times PO$ $(= AO^2 + PO^2 - 2GA \times AO) = AO^2 + \frac{3}{4}PO^2$. For AO^2 write $(AO \times PO^2)/4AS$, and if all the terms are divided by $3PO$ and multiplied by $2AS$, it will result that $\frac{4}{3}AG \times AS$ $[= \frac{1}{6}AO \times PO + \frac{1}{2}AS \times PO = (AO + 3AS)/6 \times PO = (4AO - 3SO)/6 \times PO =$ area $(APO - SPO)] =$ area APS. But GH was $3M$, and hence $\frac{4}{3}GH \times AS$ is $4AS \times M$. Therefore, the area APS that was cut off is equal to the area $4AS \times M$ that was to be cut off. Q.E.D.
>
> Corollary 1. Hence GH is to AS as the time in which the body described the arc AP is to the time in which it described the arc between the vertex A and a perpendicular erected from the focus S to the axis.

[2] Newton devised several methods for constructing equations geometrically. He derived Descartes' construction of third-degree equations in his *Lucasian Lectures on Algebra*. MP, 5, p. 489. For his early studies on the construction of third-degree equations, see MP, 2, pp. 484ff.

> Corollary 2. And if a circle ASP continually passes through the moving body P, the velocity of point H is to the velocity which the body had at the vertex A as 3 to 8, and thus the line GH is also in this ratio to the straight line which the body could describe in the time of its motion from A to P with the velocity which it had at the vertex A.[3]

11.3 Descartes' Construction of Third-Degree Equations

The Cartesian construction of third-degree equations via intersection of parabola and circle (§3.3.3) is as follows:

> Given the equation $x^3 = Px + Q$, it is required to construct it (see figure 11.2).[4]
> 1. Describe a parabola with latus rectum equal to 1 and vertex A (its equation is $y = x^2$).
> 2. Mark C on the y-axis so that $AC = (P + 1)/2$.
> 3. Draw $CE = Q/2$ horizontally from C in the direction corresponding to its sign.
> 4. Construct the circle with radius EA.[5]
> 5. The circle intersects (or touches) the parabola in at most three points. Draw perpendicular segments to the axis from each of these points.
> 6. The segments with signs as indicated by their directions, have lengths corresponding to the roots of the equation.

These Cartesian prescriptions, which were discussed by Newton in his *Lucasian Lectures on Algebra*, were well-known to seventeenth-century mathematicians. Descartes' construction of a third-degree equation leads to a geometrical diagram identical with the one employed by Newton in Proposition 30 (compare figure 11.1 with figure 11.2). Such similarity is not by chance and, I am sure, any numerate contemporary of Newton would have seen the connection with Descartes' procedure.

11.4 The Analysis behind Proposition 30, Book 1

The analysis concealed in Proposition 30 can now be considered. Indeed, Newton gave the Cartesian construction of a third-degree equation. Using a notation fully accessible to Newton—actually widely employed by him in his *Lucasian Lectures on Algebra*—denote the coordinates of the moving point P with $AO = y$ and $OP = x$.

---

[3] Newton, *Principles*, p. 510. See figure 11.3.

[4] Descartes had previously shown how to reduce the equation so that the term in x^2 is removed.

[5] Note that EA is equal to $\sqrt{\frac{1}{4}(1+P)^2 + \frac{1}{4}Q^2}$.

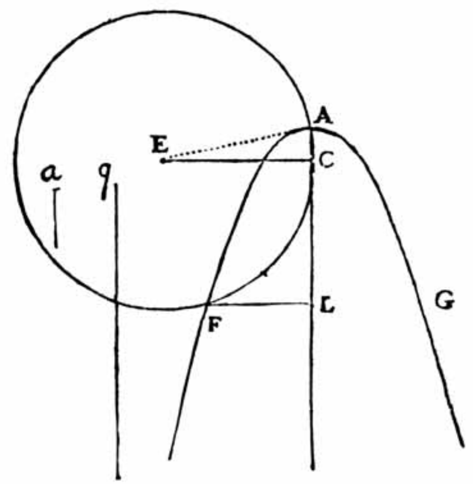


Figure 11.2
Descartes' diagram for the construction of the roots of a third-degree equation. Source: Descartes, *Géométrie* (1637), p. 205. Courtesy of the Biblioteca Angelo Mai (Bergamo).

Use the parabola's latus rectum as unit, that is, let $4AS = 1$. Thus, the parabola's equations is

$$y = x^2. \tag{11.1}$$

The area of the focal sector ASP is

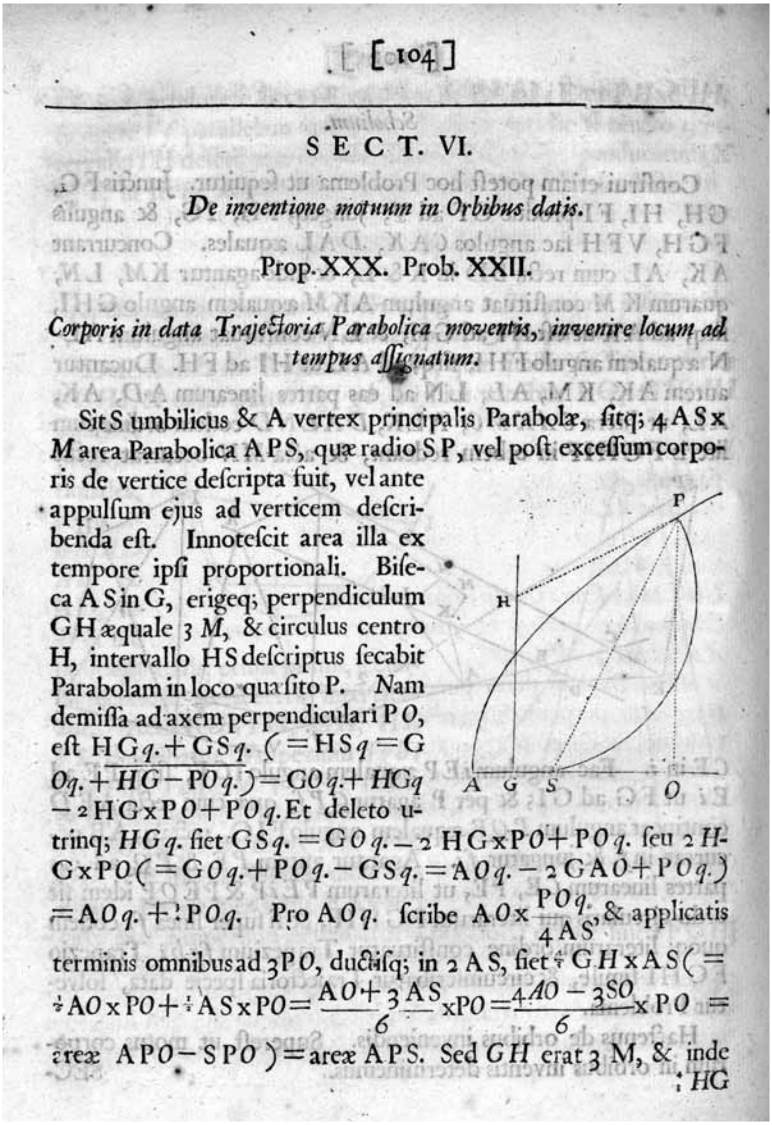
$$ASP = \frac{2}{3}xy - \frac{1}{2}x\left(y - \frac{1}{4}\right). \tag{11.2}$$

Since the area law holds,

$$ASP = \frac{h}{2}t, \tag{11.3}$$

where h is the magnitude of angular momentum per unit mass, determined by the given initial conditions (in $t = 0$, the initial velocity v_0 at the vertex A is known, and the distance from the force center S is—in our units—equal to $1/4$; therefore the magnitude of angular momentum is $h = v_0/4$). After some manipulation,

$$\frac{h}{2}t = \frac{1}{6}xy + \frac{1}{8}x = \frac{1}{6}x^3 + \frac{1}{8}x. \tag{11.4}$$



[104]

SECT. VI.

De inventione motuum in Orbibus datis.

Prop. XXX. Prob. XXII.

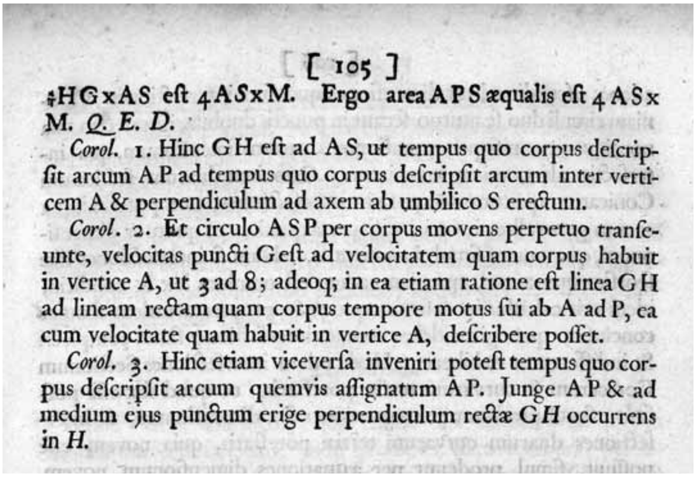
Corporis in data Trajectoria Parabolica moventis, invenire locum ad tempus aſſignatum.

Sit S umbilicus & A vertex principalis Parabolæ, ſitq; $4AS \times M$ area Parabolica APS, quæ radio SP, vel poſt exceſſum corporis de vertice deſcripta fuit, vel ante appulſum ejus ad verticem deſcribenda eſt. Innoteſcit area illa ex tempore ipſi proportionali. Biſeca AS in G, erigeq; perpendiculum GH æquale $3M$, & circulus centro H, intervallo HS deſcriptus ſecabit Parabolam in loco quæſito P. Nam demiſſa ad axem perpendiculari PO, eſt $HGq. + GSq. \;(= HSq = GOq. + \overline{HG - PO}q.) = GOq. + HGq - 2HG \times PO + POq.$ Et deleto utrinq; $HGq.$ fiet $GSq. = GOq. - 2HG \times PO + POq.$ ſeu $2HG \times PO \;(= GOq. + POq. - GSq. = AOq. - 2GAO + POq.) = AOq. + \frac{3}{4}POq.$ Pro $AOq.$ ſcribe $AO \times \frac{POq.}{4AS}$, & applicatis terminis omnibus ad $3PO$, ductiſq; in $2AS$, fiet $\frac{2}{3}GH \times AS \;(= \frac{1}{6}AO \times PO + \frac{1}{2}AS \times PO = \frac{AO + 3AS}{6} \times PO = \frac{4AO - 3SO}{6} \times PO =$ areæ $APO - SPO) =$ areæ APS. Sed GH erat $3M$, & inde

HG

Figure 11.3

Proposition 30, Book 1, as it appeared in the first edition of the *Principia*. The text of the first edition is marred by mistakes and typos that underwent several corrections. The numerous variants are edited in Newton, *Principia*, p. 187. In my analysis I use the English translation from the third edition. Source: Newton, *Philosophiae Naturalis Principia Mathematica* (1687), p. 104. Courtesy of the Biblioteca Angelo Mai (Bergamo).


[105]
¾HG×AS eſt 4ASxM. Ergo area APS æqualis eſt 4ASx M. *Q. E. D.*

Corol. 1. Hinc GH eſt ad AS, ut tempus quo corpus deſcripſit arcum AP ad tempus quo corpus deſcripſit arcum inter verticem A & perpendiculum ad axem ab umbilico S erectum.

Corol. 2. Et circulo ASP per corpus movens perpetuo tranſeunte, velocitas puncti G eſt ad velocitatem quam corpus habuit in vertice A, ut 3 ad 8; adeoq; in ea etiam ratione eſt linea GH ad lineam rectam quam corpus tempore motus ſui ab A ad P, ea cum velocitate quam habuit in vertice A, deſcribere poſſet.

Corol. 3. Hinc etiam viceverſa inveniri poteſt tempus quo corpus deſcripſit arcum quemvis aſſignatum AP. Junge AP & ad medium ejus punctum erige perpendiculum rectæ GH occurrens in H.

Figure 11.4
Corollaries to Proposition 30, Book 1. Source: Newton, *Philosophiae Naturalis Principia Mathematica* (1687), p. 105. Courtesy of the Biblioteca Angelo Mai (Bergamo).

Given the time t and v_0, one can determine x (hence, the position P of the body) by solving a third-degree equation. While the Kepler problem for ellipses is nonalgebraic (§13.3), in the case of a cometary parabolic path one is confronted with a much simpler problem. Indeed, the position is given by the solution of a third-degree algebraic equation.

Applying Descartes' rules for the construction of third-degree equations (§11.3), one can recover Newton's construction.

Rewrite equation (11.4) as

$$x^3 = -\frac{3}{4}x + 3ht, \tag{11.5}$$

and obtain $x^3 = Px + Q$, where $P = -3/4$ and $Q = 3ht$.

Then,

1. Describe the parabola $y = x^2$ with latus rectum $4AS = 1$ and vertex A (see figure 11.1).
2. Mark G on the y-axis so that $AG = (P+1)/2 = 1/8$. Hence, G is the midpoint of AS.
3. Draw $GH = Q/2 = (3/2)ht = (3/2)(v_0/4)t = (3/8)v_0 t$ horizontally from G.
4. Construct the circle with radius $HA = HS$.[6]

[6] The circle employed for the construction of a third-degree equation has to pass through the

5. The circle intersects the parabola in one point P.
6. The intersection between the circle and the parabola geometrically determines the real root $x = OP$ of equation (11.5) for any time t and thus, as required, determines the position of the body.

Thus, Newton's construction has been achieved by applying Descartes' rules.

11.5 Concluding Remarks

The demonstration of Proposition 30 (Problem 22) is geometrical. It would be erroneous, however, to think that algebraic analysis plays no role in it. Indeed, there is no doubt that Newton provided a standard construction of third-degree algebraic equations through the intersection of conics (parabola and circle) n the printed text of the *Principia*. These constructions were quite familiar to him and to most of his readers. Newton was also adamant in thinking that the composition (synthesis) should bear no trace of the resolution (analysis) carried out thanks to the manipulation of algebraic equations. Equations, he frequently stated, should be neglected in the compositive stage; they are not worthy of being published. In the *Principia*, Newton deliberately followed his own methodological prescriptions.

vertex A; thus the circle's radius is $HA = HS$.

12 Hidden New Analysis

> The second treatise [*De Quadratura*] will contain his [Newton's] Method of Quadratures ... on these [quadratures] depend certain more abstruse parts in his philosophy as hitherto published, such as Corollary 3, Proposition 41 and Corollary 2, Proposition 91.
>
> —David Gregory, 1694

> When he had reduced a proposition to the quadrature, for example, of a curve, he had completed what he considered as strictly belonging to the general Principles of Natural Philosophy; he had brought his conclusion into a tangible form, and had attained a resting place for the minds of his readers. To those few, who were then able to proceed to the numerical determinations, there could be no difficulty in translating the geometrical results into algebraic formulae.
>
> —Stephen P. Rigaud, 1838

12.1 *Concessis curvilinearum figurarum quadraturis*

Proposition 30, Book 1, tackles a rather elementary problem, one that can be resolved via common analysis. Is there any trace of the use of new analysis in the *Principia*? Indeed, there are some occurrences of application of the method of series and fluxions.[1] In this chapter I give two examples from which it will be clear that Newton deployed rather advanced quadrature techniques. Corollary 3, Proposition 41, Book 1, and Corollary 2, Proposition 91, Book 1, are analyzed in some detail.

As mentioned in section 10.2.7, point 3a, there are a number of propositions that begin with the statement that a method for squaring curvilinear figures is available. As Newton wrote in the statement of these propositions, the demonstration rests upon the conjecture that a method for squaring curves is given (*concessis curvilinearum figurarum quadraturis*). In these propositions Newton reduced the problem to the calculation of the area bounded by a curve. In subsequent corollaries or scholia he applied the general solution to particular cases (typically specifying the force law); one must therefore calculate the quadrature for specific cases. The solutions in the corollaries and scholia depend upon such calculations, but no details

Epigraphs sources: (1) Edinburgh University Library, MS Gregory C42. David Gregory's memorandum (July 1694) of a May 1694 visit to Newton. Translation in *Correspondence*, 3, p. 386. (2) Rigaud, *Historical Essay* (1838), pp. 24–5.

[1] They are analyzed in Guicciardini, *Reading the Principia* (1999).

are given on how to perform them. A famous example is the determination of the solid of least resistance (Scholium to Proposition 34, Book 2).[2] Yet this example is not unique. When Newton's contemporaries encountered these puzzling gaps in the demonstrations of the *Principia*, it was evident to them that the author had a method for squaring curves, since this method was explicitly referred to in the propositions and was used in order to reach the results in the corollaries and scholia. Like the readers of the *Enumeratio*, they often complained about Newton's reticence in revealing the analysis; if they could, they approached the master in order to obtain illumination. There are some documents in which Newton replied to such queries, giving the analytical details to David Gregory and Roger Cotes.[3] But I now turn to some of the examples of the use of new analysis in the *Principia*.

12.2 The Inverse Problem of Central Forces

The inverse problem of central forces can be stated as follows. When the centripetal force is known, and the initial position and velocity of a body acted upon by such a force are given, determine the orbit. The force is known when the force center C and the spatial dependence of the force's strength are known. The orbit is determined when the plane curve along which the body moves (the trajectory) and the position in function of time of the body are found.

In Proposition 41, Book 1, Newton reduced the inverse problem of central forces to quadratures (integrations). Proposition 41 delineates a general strategy that can be followed in order to solve the problem for any central force.

In Corollary 3, Proposition 41, Newton considered the case of an inverse-cube force. In the printed text of the *Principia* he geometrically constructed two orbits that solved the problem for an inverse-cube force. But he did not explain how the necessary quadratures (required by the general strategy applied to the case of an inverse-cube force) could be performed. Let us devote some attention to Proposition 41 and its third corollary.

12.2.1 Proposition 41, Book 1

Newton considered (figure 12.1) a body fired at V along a direction perpendicular to CV and with a given initial speed. The body is acted upon by a centripetal force directed toward force center C. Proposition 41 states,

> Supposing a centripetal force of any kind and granting the quadratures of curvilinear figures, it is required to find the trajectories in which bodies will move and also the times of their motions in the trajectories so found.[4]

[2] Proposition 35 in the first edition.

[3] See Guicciardini, *Reading the Principia* (1999), for a discussion of this topic.

[4] *Principia*, p. 218. *Principles*, p. 529.

Let VIk be the curve sought, and let IK be an infinitesimal arc traversed by the body in an infinitesimal interval of time.[5] Further, KN is drawn perpendicularly to CI. Since Kepler's area law holds, the infinitesimal interval of time is proportional to the area of the sector CIK, that is, to $(CI \times KN)/2$. Newton denoted the constant areal velocity by $Q/2$, so that the infinitesimal arc IK is traversed in the infinitesimal interval of time $(CI \times KN)/Q$. The speed v along IK can be assumed constant. It is equal to the arc IK divided by the time:

$$v = \frac{IK \times Q}{CI \times KN}. \tag{12.1}$$

Note that on the right in figure 12.1 Newton added the curves that must be squared in order to solve the inverse problem. The curve BFG represents the strength of force as a function of radial displacement (DF is thus proportional to the force's strength at I).

In Propositions 39 and 40, Book 1, Newton proved that the square of the speed at I is proportional to the area of the surface $ABFD$ under the curve that represents force's strength as a function of radial displacement CI. Newton's result, which he

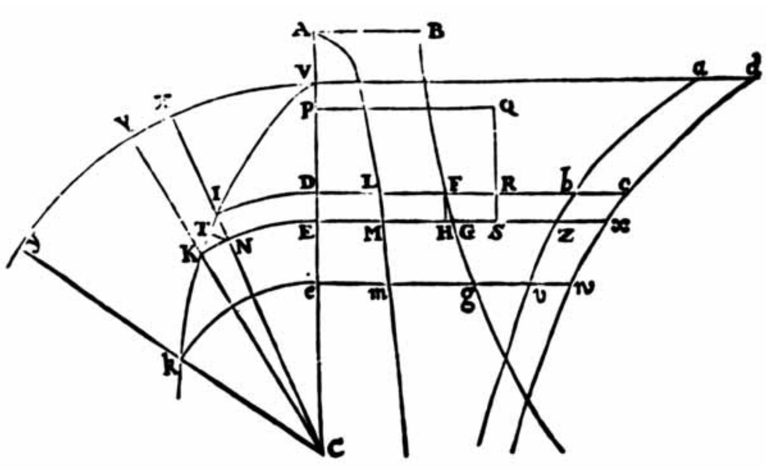


Figure 12.1
Diagram for Proposition 41, Book 1. Source: Newton, *Philosophiae Naturalis Principia Mathematica* (1687), p. 128. Courtesy of the Biblioteca Angelo Mai (Bergamo).

[5] "Let the points I and K be very close indeed to each other." *Principles*, p. 530. "Sint autem puncta I & K sibi invicem vicinissima." *Principia*, p. 219.

stated in connected prose, can be rendered in a somewhat more algebraicized way[6] as

$$v \propto \sqrt{ABFD}. \tag{12.2}$$

Thus,

$$\frac{IK \times Q}{CI \times KN} \propto \sqrt{ABFD}. \tag{12.3}$$

Newton set $CI = A$ and $Q/A = Z$, and stated[7] that

$$\frac{IK}{KN} \propto \frac{\sqrt{ABFD}}{Z}. \tag{12.4}$$

The proportion (12.4) is the basic result that allowed Newton to tackle the inverse problem of central forces in Proposition 41.[8]

Newton always had in mind the need for geometrical representability and introduced an auxiliary circle $VXYR$. From (12.4), after some simple manipulations with ratios, he deduced that

$$\text{area } CVI \left[= \frac{Q}{2} t \right] = \text{area under curve } abz = VabD, \tag{12.5}$$

---

[6] Propositions 39 and 40 can be read in modern terms as the expression of the work-energy theorem. Newton did not have the concepts of work and energy. Recall that I use $\propto$ for Newton's "is proportional to".

[7] I am here using a somewhat more symbolical formulation. What Newton actually wrote is "$\sqrt{ABFD}$ will be to Z as IK to KN." *Principles*, p. 530.

[8] In order to facilitate the understanding of the geometrical proportionality (12.4), I translate it into more familiar Leibnizian symbolic terms. Substitute $Q = h$, $IK = \mathrm{d}s$, $IN = -\mathrm{d}r$, $KN = r\mathrm{d}\theta$, $CI = r$, $CV = r_0$, and furthermore, following Propositions 39 and 40, put $ABFD = v_0^2 + 2\int_r^{r_0} F\mathrm{d}r = v^2$, where v is the speed at I, v_0 the initial speed at V. I assume mass $m = 1$. From (12.4),

$$\frac{\mathrm{d}s^2(= \mathrm{d}r^2 + r^2\mathrm{d}\theta^2)}{r^2\mathrm{d}\theta^2} = \frac{r^2}{h^2}\left(v_0^2 + 2\int_r^{r_0} F\mathrm{d}r\right).$$

By means of simple algebraic manipulations, polar differential equations for the orbit are obtained:

$$\mathrm{d}t = \frac{-\mathrm{d}r}{\sqrt{\left(v_0^2 + 2\int_r^{r_0} F\mathrm{d}r - h^2/r^2\right)}}$$

and

$$\mathrm{d}\theta = \frac{-h\mathrm{d}r}{r^2\sqrt{\left(v_0^2 + 2\int_r^{r_0} F\mathrm{d}r - h^2/r^2\right)}},$$

Note that this translation, which is close to what Johann Bernoulli did in 1710, while it is helpful for the modern reader, does not yield correctly what can be read in the *Principia* because Newton always had in mind the need to retranslate algebraic reasoning into geometry.

where the ordinate of the curve abz is

$$Db = \frac{Q}{2\sqrt{(ABFD - Q^2/A^2)}}. \tag{12.6}$$

Furthermore,

$$\text{area } CVX \left[= \frac{CX^2}{2}\theta \right] = \text{area under curve } dcx = VdcD, \tag{12.7}$$

where the ordinate of the curve dcx is

$$Dc = \frac{Q \times CX^2}{2A^2\sqrt{(ABFD - Q^2/A^2)}}. \tag{12.8}$$

If one squares abz, the functional dependence of time $[t]$ with distance $[r]$ is given.[9] If one squares acx, the functional dependence of polar angle $[\theta]$ with distance $[r]$ is given. Note that instead of working directly with polar coordinates, as in the Leibnizian translation shown in footnote 8 and in square brackets, Newton related the distance CI to geometrical quantities (areas CVI and CVX) proportional to time $[t]$ and to polar angle $[\theta]$, respectively, which are visualized in figure 12.1.[10]

12.2.2 Corollary 3, Proposition 41, Book 1: The Synthetic Construction

The inverse problem for inverse-cube forces is faced in Corollary 3, Proposition 41, Book 1.[11] The problem is how to determine the orbit of a body accelerated by a central force that varies inversely as the cube of the distance. In the *Principia*, Newton gave only the synthetic construction; he identified two orbits that answer the problem. He restricted his attention to the case in which the body is fired at V along a direction perpendicular to CV. In modern notation (see the discussion of formula (12.18)), one can say that Newton's solution corresponds to $1/r = C\sin[\gamma(\theta - \theta_0)]$ (valid for a repulsive force) and $1/r = C\cosh[\gamma(\theta - \theta_0)]$ (valid for an attractive force), where r and θ are the polar coordinates of the orbit.

The general strategy delineated in Proposition 41 can be applied to this case provided one is able to perform the necessary quadratures of the curves abz and dcx whose ordinates are (12.6) and (12.8). Newton was able to obtain these quadratures. The quadrature of (12.6) is elementary, and equation (12.8) applied to an inverse-cube force leads to the problem of squaring a curve contemplated by Newton in

[9] In order to facilitate the understanding I have added in square brackets symbolic expressions in terms of polar coordinate θ and time t that do not appear in the *Principia*.

[10] The role of visualization has been underlined by Erlichson in "The Visualization of Quadratures in the Mystery of Corollary 3 to Proposition 41 of Newton's *Principia*" (1994).

[11] On this corollary, see Brackenridge, "Newton's Easy Quadratures" (2003).

his catalogues of curves (1671) (§8.4). In the *Principia*, Newton did not perform these quadratures explicitly but simply stated the result that follows from them (figure 12.2). Corollary 3 is as follows:

> If with center C and principal vertex V, any conic VRS is described, and from any point R of it the tangent RT is drawn so as to meet the axis CV, indefinitely produced, at point T; and joining CR there is drawn the straight line CP, which is equal to the abscissa CT and makes an angle VCP proportional to the sector VCR; then, if a centripetal force inversely proportional to the cube of the distance of places from the center tends towards the center C, and the body leaves the place V with the proper velocity along a line perpendicular to the straight line CV, the body will move forward in the trajectory VPQ, which point P continually traces out; and therefore, if the conic VRS is a hyperbola, the body will descend to the center. But if the conic is an ellipse, the body will ascend continually, and will go off to infinity.
>
> And, conversely, if the body leaves the place V with any velocity and, depending on whether the body has begun either to descend obliquely to the center or to ascend obliquely from it, the figure VRS is either a hyperbola or an ellipse, the trajectory can be found by increasing or diminishing the angle VCP in some given ratio. But also if the centripetal force is changed into a centrifugal force, the body will ascend obliquely in the trajectory VPQ, which is found by taking the angle VCP proportional to the elliptic sector VCR, and by taking the length CP equal to the length CT, as above. All this follows from the foregoing proposition [41], by means of the quadrature of a certain curve, the finding of which, as being easy enough, I omit for the sake of brevity.[12]

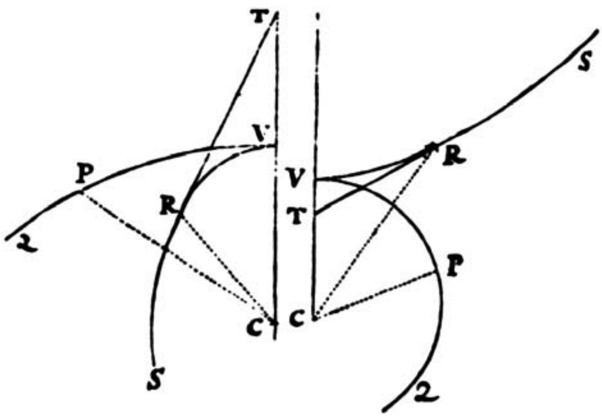


Figure 12.2
Diagram for Corollary 3, Proposition 41, Book 1. Source: Newton, *Philosophiae Naturalis Principia Mathematica* (1687), p. 130. Courtesy of the Biblioteca Angelo Mai (Bergamo).

[12] *Principia*, pp. 222–3. *Principles*, pp. 531–2.

This is the construction of the two orbits for an inverse-cube force that Newton proposed in Corollary 3, when the body is fired from V at right angles with CV. The sought orbit VPQ is a plane curve traced by the point P whose location is given in terms of two prescriptions:

1. The distance CP from force center is equal to CT.
2. The polar angle VCP is proportional to the area of the conic sector VCR.

These prescriptions refer to an auxiliary conic VRS, with vertex V and center C. The point R flows along the arc of the conic VRS. Therefore, for instance, in the diagram to the right in figure 12.2, $CP = CT$ tends to zero as the polar angle $VCP \propto VCR$ increases, so that the body spirals toward the force center.

The historian of mathematics cannot avoid a smile while reading Newton's conclusion to Corollary 3. The quadrature required was by no means easy. In what follows I consider the corollary applied to an attractive inverse-cube force and leave it to the reader to study the analogous case corresponding to a repulsive force.

12.2.3 The Hidden Analysis in Corollary 3, Proposition 41, Book 1

In 1694, David Gregory asked Newton to explicate the analysis hidden behind the synthetic construction proposed in Corollary 3. Newton replied as follows (see figure 12.3).[13] In the letter to Gregory he put the distance from force center as $CI = A = x$ (see figure 12.1), and assumed the force to be

$$F = a^4/x^3, \tag{12.9}$$

where a is a constant. From Propositions 39 and 40 it is known that the square of speed at distance x is proportional to the area $ABFD$. For an inverse-cube force such as (12.9) Newton deduced

$$ABFD = 2a^4/x^2 - 2a^4/c^2. \tag{12.10}$$

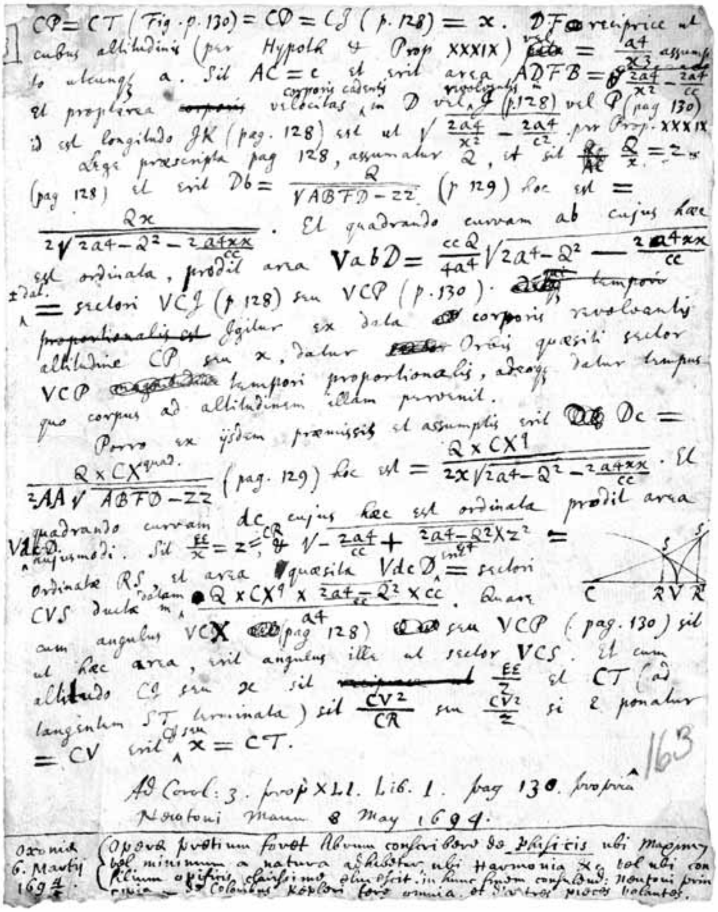
Note that this result has the wrong factor 2 at the numerator.[14]

Newton then stated that Corollary 3 is solved by squaring the curves abz and dcx, whose ordinates Db and Dc are given by equations (12.6) and (12.8) for $F = a^4/x^3$. That is,

$$Db = \frac{Q}{2\sqrt{(2a^4 - Q^2)/x^2 - 2a^4/c^2}} = \frac{1}{2}\frac{Qx}{\sqrt{2a^4 - Q^2 - 2a^4x^2/c^2}}, \tag{12.11}$$

[13] *Correspondence*, 6, pp. 435–7.

[14] In modern notation, note that c is chosen so that $-2a^4/c^2 = -2a^4/x_0^2 + v_0^2$, where x_0 and v_0 are the initial position and velocity.



CP = CT (Fig. p. 130) = CD = CI (p. 128) = x. DF reciproce ut cubus altitudinis (per Hypoth & Prop XXXIX) vel $= \frac{a^4}{x^3}$ assumpto utcunque a. Sit AC = c et erit area $ADFB = \frac{2a^4}{x^2} - \frac{2a^4}{cc}$ et propterea corporis cadentis velocitas in D vel I (p128) vel P (pag 130) id est longitudo IK (pag. 128) erit ut $\sqrt{\frac{2a^4}{x^2} - \frac{2a^4}{c^2}}$ per Prop. XXXIX. Lege praescripta pag 128, assumatur Q, et sit $\frac{Q}{x} = Z$ (pag 128) et erit $Db = \frac{Q}{\sqrt{ABFD - zz}}$ (p 129) hoc est $= \frac{Qx}{2\sqrt{2a^4 - Q^2 - \frac{2a^4xx}{cc}}}$. Et quadrando curvam ab cujus haec est ordinata, prodit area $VabD = \frac{ccQ}{4a^4}\sqrt{2a^4 - Q^2 - \frac{2a^4xx}{cc}}$ = sectori VCI (p 128) seu VCP (p. 130) tempori dato. Igitur ex data corporis revolventis altitudine CP seu x, datur Orbis quaesiti sector VCP tempori proportionalis, adeoque datur tempus quo corpus ad altitudinem illam pervenit.

Porro ex ijsdem praemissis et assumptis erit $Dc = \frac{Q \times CX^{quad.}}{2AA\sqrt{ABFD - zz}}$ (pag. 129) hoc est $= \frac{Q \times CX^q}{2x\sqrt{2a^4 - Q^2 - \frac{2a^4xx}{cc}}}$. Et quadrando curvam dc cujus haec est ordinata prodit area VdcD ejusmodi. Sit $\frac{EE}{x} = z$ [illegible] ordinatae RS, et area quaesita VdcD = sectori CVS ducto in [illegible]. Quare cum angulus VCX (pag 128) seu VCP (pag. 130) sit ut haec area, erit angulus ille ut sector VCS. Et cum altitudo CI seu x sit $\frac{EE}{Z}$ et CT (ad tangentem ST terminata) sit $\frac{CV^2}{CR}$ seu $\frac{CV^2}{Z}$ si Z ponatur = CV erit CI seu x = CT.

163

Ad Coroll: 3. prop XLI. Lib. 1. pag 130. propria Newtoni manu 8 May 1694.

Oxoniae 6. Martij 169 4/5. [illegible]

Figure 12.3
An example of how information about the analytical subtext of the *Principia* circulated among Newton's acolytes. Manuscript sent by Newton to David Gregory upon his request in 1694. It contains the analytical details of the quadratures required in Corollary 3, Proposition 41, Book 1, of the *Principia.* See Guicciardini, *Reading the Principia* (1999), pp. 216–23, and especially Erlichson, "The Visualization of Quadratures in the Mystery of Corollary 3 to Proposition 41 of Newton's *Principia*" (1994) and Brackenridge, "Newton's Easy Quadratures" (2003). Source: Royal Society: Greg.(ory) M.S. Folio 163. By permission of the Royal Society.

and (figure 12.4)

$$Dc = \frac{Q \times CX^2}{2x^2\sqrt{(2a^4 - Q^2)/x^2 - 2a^4/c^2}} = \frac{Q \times CX^2}{2x\sqrt{2a^4 - Q^2 - 2a^4x^2/c^2}}. \qquad (12.12)$$

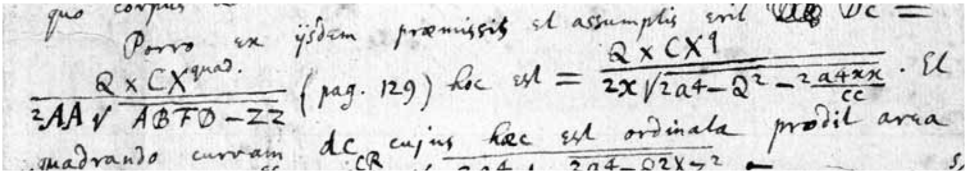


Figure 12.4
Detail of Newton's reply to Gregory. Here Newton translated one of the basic proportionalities stated in Proposition 41 into symbols (equation 12.12). Source: Royal Society: Greg.(ory) M.S. Folio 163. By permission of the Royal Society.

The quadrature of (12.11) is elementary. Newton noted that the area $VabD$,[15] which gives the functional dependence of time with distance $CI = x$ from the force center, is

$$VabD = \frac{c^2 Q}{4a^4}\sqrt{2a^4 - Q^2 - 2a^4x^2/c^2} \pm \text{a given constant.} \tag{12.13}$$

The quadrature of (12.12) is more difficult and must have created some problems for Gregory.

In order to square the curve dcx Newton must have made recourse to his second catalogue of curves. By inspection of the first column $(d/(z\sqrt{e+fz^\eta}) = y)$ in figure 12.5 we see that the quadrature of (12.12) is reduced to the first case of the seventh order for $z \to x$, $\eta = 2$, $d = (Q/2) \times CX^2$, $e = 2a^4 - Q^2$, $f = -2a^4/c^2$. Note that the abscissa of the curve to be squared is x in the letter to Gregory and z in the 1671 table (hence, $z \to x$).[16]

---

[15] The fact that Newton did not have an integral sign and did not express the limits of integration creates some problems for the modern reader. Newton here calculated, for $x_0 > x_1$,

$$VabD = \frac{Q}{2}t = \frac{Q}{2}\int_{x_0}^{x_1} \frac{x\,dx}{\sqrt{2a^4 - Q^2 - 2a^4x^2/c^2}}$$

and

$$VdcD = \frac{CX^2}{2}\theta = \frac{CX^2}{2}\int_{x_0}^{x_1} \frac{Q\,dx}{x\sqrt{2a^4 - Q^2 - 2a^4x^2/c^2}}.$$

[16] It is useful to translate the first case of the seventh order into Leibnizian notation. For $\eta = 2$, Newton evaluated the integral $\int \delta/(z\sqrt{e+fz^2})\,dz$ (δ, e, f constants). By substitution of variables $z = x^{-1}$, he reduced it to the conic area $s = \int v dx = \int \sqrt{f + ex^2}\,dx$. Namely,

$$\int \frac{\delta}{z\sqrt{e+fz^2}}\,dz = \frac{2\delta}{f}\left|\frac{1}{2}xv - s\right| + C = \frac{2\delta}{f}\left|\frac{1}{2}x\sqrt{f+ex^2} - \int \sqrt{f+ex^2}\,dx\right| + C = t.$$

Verify by differentiation that

$$\frac{dt}{dz} \quad = \quad \frac{dt}{dx}\frac{dx}{dz} = \frac{2\delta}{f}\left(\frac{1}{2}v + \frac{1}{2}x\frac{dv}{dx} - v\right)\frac{dx}{dz} = \frac{2\delta}{f}\left(-\frac{1}{2}\sqrt{f+ex^2} + \frac{ex}{2\sqrt{f+ex^2}}\right)\frac{dx}{dz}$$

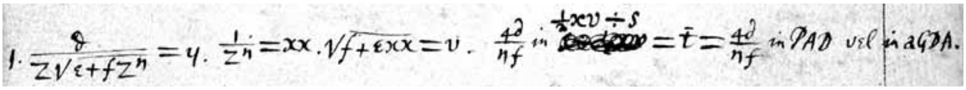


Figure 12.5

First case of the seventh order of the second catalogue of curves in the manuscript of *De Methodis* (1671). Note that d is a constant, not a differential. Also, Newton did not use the modern symbol for the absolute value $|\frac{1}{2}xv - s|$ but rather one that he found in Barrow's works. Newton wrote $\div$ for "the Difference of two Quantities, when it is uncertain whether the latter should be subtracted from the former, or the former from the latter." (Newton, *Two Treatises* (1745), p. 25). Thus, Newton wrote $\frac{1}{2}xv \div s$. Source: Add. 3960.14, f. 81. Reproduced by kind permission of the Syndics of Cambridge University Library.

The substitution of variables in the second and third columns must be slightly modified. Further, a complication arises from the fact that the auxiliary conic is lettered differently (compare figures 12.6 and 12.7). Recall that the polar angle of the trajectory is proportional to the conic sector CVS.

Newton set

$$z = \frac{\epsilon^2}{x} = CR, \tag{12.14}$$

where $\epsilon = CV$, and (see third column)

$$v = \sqrt{-2a^4c^{-2} + (2a^4 - Q^2)\epsilon^{-4}z^2} = RS. \tag{12.15}$$

The sought area $VdcD$ given in the fourth column is

$$VdcD = \frac{2d}{f}\left|\frac{1}{2}zv - s\right| = t, \tag{12.16}$$

where s is the area of the surface subtended under the conic whose ordinate is $RS = v$ and abscissa $CR = z$ (see figure 12.6).[17]

Gregory inserted this solution into his manuscript "Notae" in a space that he had left empty in his running commentary on the *Principia*.[18] Newton had clearly given him some much-needed assistance. It has often been repeated that Newton

$$= \frac{-\delta}{\sqrt{f + ex^2}}\frac{dx}{dz} = \frac{-\delta z}{\sqrt{fz^2 + e}}\frac{-1}{z^2} = \frac{\delta}{z\sqrt{fz^2 + e}}.$$

[17] The fact that Newton chose this more circuitous substitution of variables instead of putting $d = Q\epsilon^2/2$ reveals his intention of interpreting geometrically x and z as lengths of segments. Indeed the constant ϵ corresponds to the length of CV, that is, the distance of the center from the principal vertex of the conic used in the construction of Corollary 3. Newton deployed algebra having in mind the necessity of interpreting geometrically the symbols that he manipulated.

[18] For Gregory, see Eagles, The Mathematical Work of David Gregory (1977). David Gregory began to read the *Principia* carefully in the early autumn of 1687. The result of his careful analysis

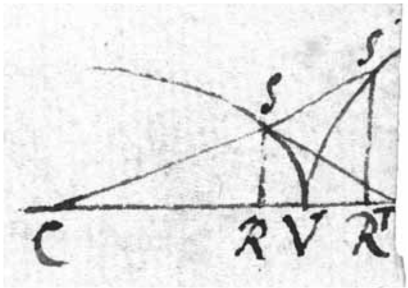


Figure 12.6
Detail of Newton's reply to Gregory. The conics required in the synthetic construction. Source: Royal Society: Greg.(ory) M.S. Folio 163. By permission of the Royal Society.

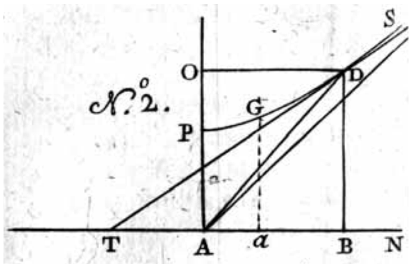


Figure 12.7
The quadrature of the first case of the seventh form is reduced to the quadrature of a conic area. This figure precedes the second catalogue of curves. This is the conic referred to in the third column in figure 12.5 and coincides (apart from different lettering) with the conic in Newton's reply to Gregory. The sector CVS in the letter to Gregory coincides with sector APD in *De Methodis*. Source: Newton, *Opuscula Mathematica, Philosophica et Philologica* (1744), 1, Tab. V. Courtesy of the Biblioteca Angelo Mai (Bergamo).

was unable to write the differential equations of motion. The case of Corollary 3 to Proposition 41 proves that this widely accepted judgment needs to be better qualified.

is a 213-page-long manuscript entitled "Notae in Newtoni Principia Mathematica Philosophiae Naturalis." Gregory's "Notae" are extremely important because they reveal how a well-trained mathematician could understand Newton's *Principia*. Despite this, they have been analyzed only superficially by historians of science. The original manuscript in Gregory's hand is kept in the Royal Society (London), shelved as MS 210. The first 30 pages, a commentary on the first nine sections of Book 1, are dated from September 1687 to April 1688. The remaining pages are dated from December 1692 to January 1694. There are also later additions written on slips of paper affixed by paste or wax. The last addition was made in 1708. There are also three transcripts: in Christ Church (Oxford), in the University Library (Edinburgh), and in the Gregory Collection of the University of Aberdeen. Royal Society Library, MS 210. Christ Church Library. University Library (Edinburgh), MS DC.4.35. University of Aberdeen Library, MS 465.

12.2.4 The Construction of Corollary 3, Proposition 41

Now the synthetic construction proposed in the *Principia* in Corollary 3, Proposition 41, should be clear. The area $VdcD$ (see equation (12.16)), which gives the functional dependence of polar angle VCP with distance CP, is proportional to the area obtained by subtracting the area of the hyperbolic sector $VSR = s$ from the triangular area $CRS = zv/2$; therefore it is proportional to the area of half the "hyperbolic angle" CVS (see figure 12.6). In symbols,

$$VdcD = k \times CVS. \tag{12.17}$$

The polar angle of the sought trajectory is thus geometrically represented by the area of sector CVS, as stated in the printed Corollary 3.[19]

It is a simple property of the hyperbola that the tangent in a point $S(z, v)$ cuts the axis of the abscissae z in a point T such that CT is inversely proportional to $CR = z$, that is, $CT = CV^2/CR = \epsilon^2/z$.[20] Therefore, CT can be taken as a geometrical representation of the radial distance $x = CP$ from the force center, as stated in the printed Corollary 3 (compare figure 12.2 with figure 12.6).[21]

Newton's geometrical construction of the orbit for an attractive inverse-cube force corresponds to what nowadays would be expressed as

$$\frac{1}{x} = C \cosh[\gamma(\theta - \theta_0)], \tag{12.18}$$

where x and θ are the polar coordinates of the orbit. Indeed, $CR = \epsilon \cosh t$, where t is twice the area of the sector CVS.

---

[19] Note that in his reply to Gregory, Newton made a mistake in writing that the constant of proportionality is

$$k = \frac{Qc^2(2a^4 - Q^2)}{a^4};$$

it should be

$$k = \frac{2d}{f} = \frac{Qc^2}{2a^4}.$$

This mistake is interesting, since it supports a surmise that Newton was using the quadrature formula of the first case of seventh order of the second catalogue of curves. The origin of this mistake seems to reside in the fact that Newton, looking back to his catalogues composed in 1671, must have made the trivial mistake consisting in using $4ed/f$ instead of $2d/f$. The extra term $(2a^4 - Q^2)$, indeed, corresponds to the constant e.

[20] For a hyperbola with equation $z^2/\epsilon^2 - v^2/\epsilon'^2 = 1$ in Cartesian coordinates, one obtains $CT = z - v(dz/dv) = \epsilon^2/z$.

[21] Note that the printed diagram of Corollary 3 has a slightly different lettering from the one used in the letter to Gregory.

12.3 Attraction of Solids of Revolution

Propositions 90 and 91, Book 1, deal with the attraction exerted by homogeneous solids of revolution on external points situated on the prolongation of the axis of revolution.

12.3.1 Proposition 90, Book 1

In Proposition 90, Newton determined the attraction exerted by a uniform circular disk ("a circle") with center A and radius AD (see figure 12.8) on a point situated in P (PA is normal to the disk). Let PH be equal to PD, and let the ordinates of curve IKL be such that, if $PF = PE$, then FK is as "the force by which that point [E] attracts body P towards A."[22] Consider the small line Ee and the ring (the annulus) generated by rotation of Ee about the axis PAH. The strength of the attraction exerted by the ring is

$$F_{\text{ring}} \propto FK \cdot AE \cdot Ee \cdot \frac{PA}{PE}, \tag{12.19}$$

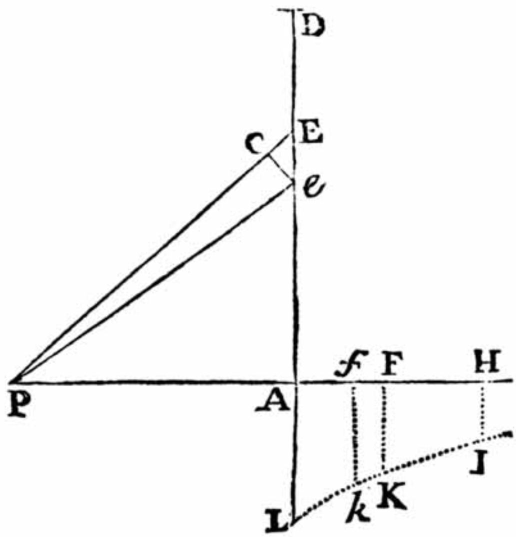


Figure 12.8
Attraction exerted by a disk. AD is the disk's radius. P is the position of a point whose attraction toward the disk must be determined. Source: Newton, *Philosophiae Naturalis Principia Mathematica* (1726), p. 214. Courtesy of the Biblioteca Angelo Mai (Bergamo).

[22] *Principia*, p. 329. *Principles*, p. 614.

since the surface of the ring is $2 \cdot \pi \cdot AE \cdot Ee$, and the cosine factor (PA/PE) is introduced by symmetry considerations. Recall that I use the symbol $\propto$ as shorthand for Newton's expression "is proportional to."

Draw eC such that $PC = Pe$. Let $Pf = Pe$ and thus $CE = Ff$. When the angle ePE tends to zero, one can state that $AE/PE = CE/Ee$ (here, as is often the case, Newton did not make explicit this limit argument but simply relied on the fact that Ee is infinitesimal).[23] Therefore (taking into consideration that $AE \cdot Ee = PE \cdot CE = PE \cdot Ff$),

$$F_{\text{ring}} \propto FK \cdot PE \cdot Ff \cdot \frac{PA}{PE} = PA \cdot FK \cdot Ff. \tag{12.20}$$

That is, the strength of the force exerted on the point situated at P by the ring generated by the revolution of Ee is proportional to "the [infinitesimal] area $FKkf$ multiplied by PA." In conclusion, the force directed toward A exerted on P by the circular disk is "as the whole area $AHIKL$ multiplied by PA."[24]

The problem faced in Proposition 90 is thus reduced to the calculation of the area of the surface $AHIKL$. In the next three Corollaries, Newton performed some exemplary quadratures. For instance, in Corollary 2 he wrote that,

> if the forces of the points at the distances D are inversely as any power D^n of the distances (that is, if FK is as $1/D^n$ [$=1/PE^n$]), and hence the area $AHIKL$ is as $1/PA^{n-1} - 1/PH^{n-1}$, the attraction of the corpuscle P toward the circle [the disk] will be as $1/PA^{n-2} - PA/PH^{n-1}$ [where $PH = PD$].[25]

This simple quadrature is achieved by application of results known in the pre-calculus period. Indeed, the area of the surface over the interval AH and under the curve whose ordinate is $FK = -k/PF^n$ is, applying a well-known law, $(k/(n-1))(1/PA^{n-1} - 1/PH^{n-1})$.[26]

12.3.2 Proposition 91, Book 1

In Proposition 91, Newton considered the attraction exerted by a homogeneous solid of revolution on an external point situated on the axis of revolution. He partitioned the solid into circular disks and obtained the total attraction by summing the component attractions of the disks. In figure 12.9 the solid is generated by the rotation of the curve DRE around the axis AB. The ordinate FK represents the strength

[23] Ee is a "linea quam minima." From Section 1, Book 1, it is known that infinitesimal arguments must be translated into the language of the method of first and ultimate ratios (§9.3).

[24] *Principia*, p. 330. *Principles*, p. 614

[25] *Principia*, pp. 330–1. *Principles*, p. 615.

[26] The law, well-known to Newton's predecessors such as Wallis (§7.1), can be written as $\int x^n = x^{n+1}/(n+1) + C$.

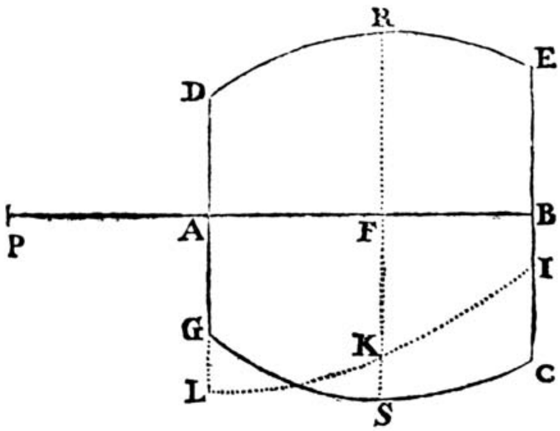


Figure 12.9
Attraction exerted by a solid of revolution on an external point situated on the axis of revolution. The solid is subdivided into disks. The disk at distance PF from the point situated at P has diameter RS. Source: Newton, *Philosophiae Naturalis Principia Mathematica* (1726), p. 216. Courtesy of the Biblioteca Angelo Mai (Bergamo).

of the attraction exerted by the disk whose diameter is RS on the "corpuscle" at P. Thus the total attraction exerted by the solid is given by the area of the surface $LABI$.

According to Corollary 2, Proposition 90, if the force is as any power of the distance, that is, as $-1/D^n$,

$$F_{\text{disk}} \propto FK = \frac{1}{PF^{n-2}} - \frac{PF}{PR^{n-1}}. \tag{12.21}$$

If the force is inverse-square ($n = 2$),

$$F_{\text{disk}} \propto FK = 1 - \frac{PF}{PR}. \tag{12.22}$$

According to Proposition 91 the attraction on a point situated on the prolongation of the axis AB at P, under the hypothesis that the mass of the solid of revolution is distributed homogeneously, is proportional to the area of the surface subtended by the curve IKL whose abscissa is PF and ordinate is FK, calculated between limits $PF = PA$ and $PF = PB$.

In Corollary 1 the solid is a cylinder and the force is inverse-square. In this case the area of the surface subtended by curve IKL is easily calculated ("which can easily be shown from the quadrature of the curve LKI").[27]

[27] "id quod ex curvae LKI quadratura facile ostendi potest." *Principia*, p. 332. *Principles*, p. 616.

12.3.3 Corollary 2, Proposition 91, Book 1: The Synthetic Construction

Corollary 2, Proposition 91, Book 1, concerns an oblate spheroid generated by rotation of ellipse $AGBC$ around the minor axis AB (figure 12.10). According to Proposition 91, the attraction on a point situated on the prolongation of the minor axis at P, under the hypothesis that the mass of the spheroid is distributed homogeneously and that the attracting force is inverse-square, is proportional to the area of the surface subtended by the curve whose abscissa is PE and ordinate is

$$F_{\text{disk}} = 1 - \frac{PE}{PD}, \tag{12.23}$$

calculated between limits $PE = PA$ and $PE = PB$. Indeed, (12.23) gives the strength of the attraction exerted by a disk whose radius is ED. The total attraction exerted by the ellipsoid is calculated by summing the component attractions of the disks into which it is sliced.

In the *Principia*, Newton could not refer the reader to his unpublished discoveries on quadratures. He thus simply gave the geometrical construction that corresponds to the quadrature of the above mentioned curve. Corollary 2, without any reference to Newton's catalogues of curves, is a complete mystery. It is as follows:

> Hence also the force becomes known by which a spheroid $AGBC$ attracts any body P, situated outside the spheroid in its axis AB. Let $NKRM$ be a conic whose ordinate ER, perpendicular to PE, is always equal to the length of the

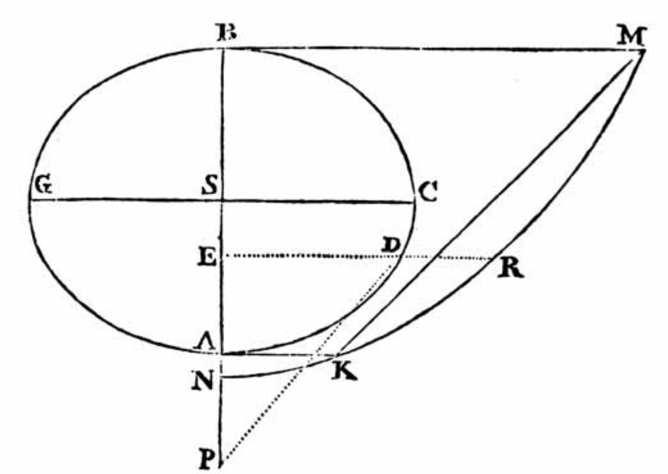


Figure 12.10
Attraction exerted by an ellipsoid on an external point. Source: Newton, *Philosophiae Naturalis Principia Mathematica* (1726), p. 217. Courtesy of the Biblioteca Angelo Mai (Bergamo).

> line PD, which is drawn to the point D in which the ordinate cuts the spheroid. From the vertices A and B of the spheroid, erect AK and BM perpendicular to the axis AB of the spheroid and equal respectively to AP and BP, and therefore meeting the conic in K and M; and join KM cutting off the segment $KMRK$ from the conic. Let the center of the spheroid be S, and its greatest semidiameter SC. Then the force by which the spheroid attracts the body P will be to the force by which a sphere described with diameter AB attracts the same body as $(AS \times CSq - PS \times KMRK)/(PSq + CSq - ASq)$ to $AScub./3PSquad$. And by the same mode of computation it is possible to find the forces of the segments of the spheroid.[28]

12.3.4 The Hidden Analysis in Corollary 2, Proposition 91, Book 1

Quite understandably, Corollary 2 aroused the interest of competent readers of the *Principia*, not least because it played a fundamental role in Newton's treatment of the earth's shape. It was, however, unclear how Newton could find the above geometrical construction. Some of Newton's acolytes, such as David Gregory and Roger Cotes, discussed this corollary with Newton and referred to the second catalogue of curves, where Newton had developed his analytical method of quadratures. Indeed, the construction is achieved by finding the area subtended to a curve (whose ordinate is equation (12.23)), and the calculation of this area is reduced to the calculation of the area subtended to the conic $NKRM$. As discussed in chapter 8, such transmutations of areas of curves to conic areas is the basic ingredient of the second catalogue of curves of *De Methodis* (§8.4.4).

There is evidence of how Newton privately discussed the hidden analytical quadratures from Gregory's manuscript commentary to the *Principia*, and from Gregory's and Cotes's correspondence with Newton. Gregory discussed this corollary with Newton in the 1690s, and was able to reproduce the instructions received from the master in his "Notae in Newtoni Principia Mathematica Philosophiae Naturalis."[29] It is interesting that the first extant letter by Cotes to Newton concerns Corollary 2, Proposition 91, Book 1. The young Plumian Professor had just been chosen, thanks to Richard Bentley's recommendation, as editor of the second edition of the *Principia*. In presenting himself to Newton, he chose this difficult Corollary as proof of his skill in mathematics. On August 18, 1709, he wrote,

> Some days ago I was examining the 2d Cor: of Prop. 91 Lib. I & found it to be true by ye Quadratures of ye 1st & 2d Curves of the 8th Form of ye second Table in Yr Treatise *De Quadrat.*[30]

---

[28] Note that CSq means CS^2, $AScub.$ means AS^3, etc. *Principia*, pp. 332–3. *Principles*, pp. 616–7. Whiteside has shown how the above geometrical construction corresponds to the result achieved via Newton's catalogues of curves of *De Methodis*. See MP, 6, p. 226.

[29] See Guicciardini, *Reading the Principia* (1999) for a discussion of this evidence.

[30] *Correspondence*, 5, p. 3.

After having circulated among Newton's acolytes, the analytical quadrature necessary for Corollary 2, Proposition 91, given in terms of the eighth form of the second catalogue of curves, was printed in the Appendix to Andrew Motte's English translation of the *Principia* (1729).[31]

Thus, rather than a reconstruction of what Newton *might* have done, there is information on how such technicalities were understood and circulated in the Newtonian circle. Reading from the Appendix to Motte's translation, one has the following.

Let the minor semi-axis $SA = r$, the major semi-axis $SC = c$, $PS = \delta$, $PB = PS + SB = \delta + r = a$, $PA = PS - SA = \delta - r = \alpha$, $PE = x$ (figure 12.11). By construction the curve $NKRM$ is defined as "a conic whose ordinate ER, perpendicular to PE, is always equal to the length of the line PD, which is drawn to the point D in which the ordinate cuts the spheroid" (§.12.2.3). Equation (12.26) proves that that $NKRM$ is a conic.

Recall (see equation (12.23)) that the force exerted on a point mass situated at P by a disk of radius ED is proportional to

$$F_{\text{disk}} = 1 - \frac{PE}{PD}. \tag{12.24}$$

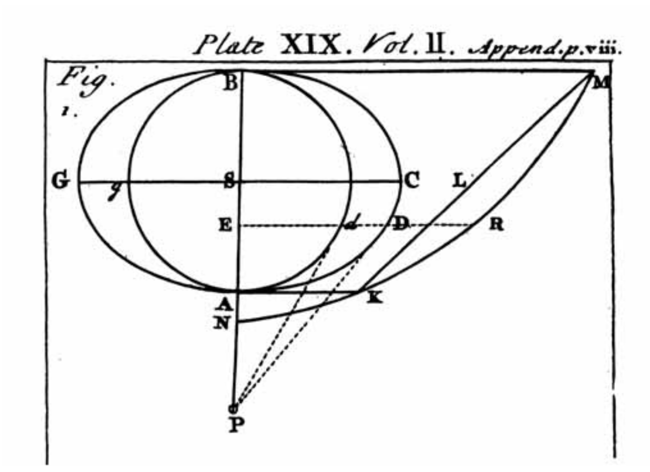


Figure 12.11
Attraction exerted by an ellipsoid. Source: Newton, *The Mathematical Principles of Natural Philosophy. Translated into English by Andrew Motte* (1729), 2: Plate 19. Courtesy of the Syndics of Cambridge University Library

[31] Entitled "Explications, (given by a Friend,) of some Propositions in this Book, not demonstrated by the Author."

Because of a well-known property of the ellipse,[32]

$$ED^2 = \frac{SC^2}{SA^2} \times AE \times EB = \frac{c^2}{r^2}(x-\alpha)(a-x) = \frac{c^2}{r^2}(-a\alpha + 2\delta x - x^2). \quad (12.25)$$

Since $PE = x$ and $PD^2 = ED^2 + PE^2$, it follows that

$$PD = ER = \sqrt{ED^2 + PE^2} = \sqrt{\frac{c^2}{r^2}(-a\alpha + 2\delta x - x^2) + x^2}. \quad (12.26)$$

Therefore,

$$F_{\text{disk}} = 1 - \frac{PE}{PD} = 1 - y = 1 - x \Big/ \sqrt{-\frac{a\alpha c^2}{r^2} + \frac{2\delta c^2}{r^2}x + \frac{r^2 - c^2}{r^2}x^2}. \quad (12.27)$$

Thus, because of (12.27), the force exerted by the ellipsoid will be given by the area t_{ell} of the surface subtended by the curve whose abscissa is x and whose ordinate is $1-y$.

By inspection of the second case of the eighth order of the second catalogue of curves of *De Methodis* (reproduced also in *De Quadratura*), one can evaluate the fluent t_{ell} (in Leibnizian terms, the integral $t_{\text{ell}} = \int(1-y)dx$) as follows (figure 12.12). The curve listed in the first column of the table is reduced to the second addendum y of the right-hand side of (12.27) for $\eta = 1$, $z = x$, $d = 1$, $e = -a\alpha c^2/r^2$, $f = 2\delta c^2/r^2$, $g = (r^2 - c^2)/r^2$. In the fourth column we read the value of the fluent t of y ($t = \int y dx$). In the second and third columns one finds the substitutions of variables.

The required fluent, t_{ell}, of $1-y$ is thus

$$t_{\text{ell}} = x - t = x - \frac{-4fs + 2fxv + 4ev}{4eg - f^2}, \quad (12.28)$$

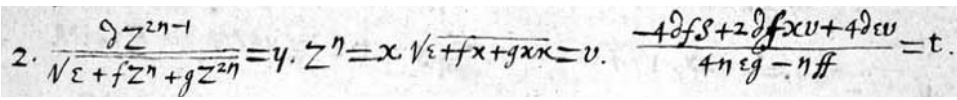


Figure 12.12

The second case of the eight order of the second catalogue of curves in the manuscript of *De Methodis* (1671). Note that d is a constant, not a differential. Source: Add. 3960.14, f. 82. Reproduced by kind permission of the Syndics of Cambridge University Library.

[32] Note that $(x-\alpha)(a-x) = xa + \alpha x - x^2 - \alpha a$, $xa + \alpha x = x(a+\alpha)$, and $a + \alpha = PB + PA = PS + SB + PS - SB = 2PS = 2\delta$.

that is,[33]

$$
\begin{aligned}
t_{\text{ell}} &= x - t \\
&= x - \left(-\frac{8\delta c^2}{r^2}s + \frac{4\delta c^2}{r^2}xv - \frac{4a\alpha c^2}{r^2}v\right) \Big/ \left(-\frac{4a\alpha c^2}{r^2}\frac{r^2-c^2}{r^2} - \frac{4\delta^2 c^4}{r^4}\right) \\
&= x + \frac{\delta xv - a\alpha v - 2\delta s}{c^2+\delta^2-r^2}. \qquad (12.29)
\end{aligned}
$$

Note from the second and third columns of Newton's catalogue that

$$
v = \sqrt{-\frac{a\alpha c^2}{r^2} + \frac{2\delta c^2}{r^2}x + \frac{r^2-c^2}{r^2}x^2} = PD = ER, \qquad (12.30)
$$

and s is the area under curve with abscissa x and ordinate v (namely, it is the conic area $AKRMB$). Recall that the area s is considered by Newton as known, and has to be evaluated by power series expansion.

This completes the analysis of the problem. Equation (12.29) is the fluent that allows resolution of the problem when (12.29) is evaluated between limits $x = PA = \alpha$ and $x = PB = a$.

When $x = a$, $v = a$; and when $x = \alpha$, $v = \alpha$. Thus, the force F exerted by the ellipsoid on P is proportional to

$$
\begin{aligned}
F = a - \alpha + \frac{\delta(a^2-\alpha^2) - a\alpha(a-\alpha) - 2\delta s}{c^2+\delta^2-r^2} &= \\
2r + \frac{2\delta^2 r + 2r^3 - 2\delta s}{c^2+\delta^2-r^2} &= \\
\frac{2rc^2 + 2r\delta^2 - 2r^3 + 2\delta^2 r + 2r^3 - 2\delta s}{c^2+\delta^2-r^2} &= \\
\frac{2rc^2 + 2\delta(2\delta r - s)}{c^2+\delta^2-r^2}&. \qquad (12.31)
\end{aligned}
$$

[33] This calculation is somewhat tedious. First, $4c^2/r^2$ is factorized, obtaining

$$
t_{\text{ell}} = x - t = x - \frac{4c^2/r^2(-2\delta s + \delta xv - \alpha av)}{4c^2/r^2(-\alpha a(r^2-c^2)/r^2 - \delta^2 c^2/r^2)} = x - \frac{-2\delta s + \delta xv - \alpha av}{-\alpha a(r^2-c^2)/r^2 - \delta^2 c^2/r^2}.
$$

The denominator can be simplified by taking into account that $a = \delta + r$ and $\alpha = \delta - r$. Therefore, $\alpha a = (\delta + r)(\delta - r) = \delta^2 - r^2$. Thus,

$$
-\alpha a(r^2-c^2)/r^2 - \delta^2 c^2/r^2 = (r^2-\delta^2)(r^2-c^2)/r^2 - \delta^2 c^2/r^2 = r^2 - c^2 - \delta^2.
$$

Note that, since $a = \delta + r$ and $\alpha = \delta - r$, one has that $\delta(a^2 - \alpha^2) - a\alpha(a - \alpha) = \delta[(\delta + r)^2 - (\delta - r)^2] - (\delta + r)(\delta - r)(\delta + r - \delta + r) = 2\delta^2 r + 2r^3$.

Newton did not have a symbol, such as $\int$, for the integral, and when he had to evaluate what in Leibnizian terms one would call a definite integral he made the limits of integration explicit in connected prose. This renders his calculations difficult to follow, especially for a reader trained in contemporary mathematics.[34]

12.3.5 The Construction of Corollary 2, Proposition 91, Book 1

But Newton needed the construction, and it is only the construction, not the analysis, that he published in the *Principia*. Let us interpret (12.31) geometrically.

Note that $2\delta r$ is the area of the trapezium $ABMK$,[35] s is the conic area $AKRMB$, and $2\delta r - s$ is the area of the segment $KMRK$ (see figure 12.11). Consequently (12.31) is interpreted geometrically as

$$\frac{2SA \times SC^2 - 2PS \times KRMK}{SC^2 + PS^2 - SA^2}, \tag{12.32}$$

as is stated in the *Principia*.

Further, Newton showed by a much simpler quadrature technique that the attraction exerted by a sphere $AdBg$ on P is $2SA^3/3PS^2$.[36] Thus, the statement follows:

[34] It is helpful to translate this calculation into more familiar Leibnizian notation. In the second case of the eighth order of the second catalogue of curves, Newton evaluated the following integral (departing here from Newton's practice, I render the constant of integration C explicit):

$$\int \frac{z^{2\eta-1}}{\sqrt{e + fz^\eta + gz^{2\eta}}} dz = \frac{-4f \int v dx + 2fxv + 4ev}{4eg - f^2} + C = t,$$

where the substitution of variable is $z^\eta = x$, $v = \sqrt{e + fz^\eta + gz^{2\eta}}$, and $s = \int v dx$.
For $\eta = 1$, one gets

$$\int \frac{xdx}{\sqrt{e + fx + gx^2}} = \frac{-4f \int \sqrt{e + fx + gx^2} dx + 2fx\sqrt{e+fx+gx^2} + 4e\sqrt{e + fx + gx^2}}{4eg - f^2} + C = t.$$

Verify by differentiation that

$$\frac{dt}{dx} = \frac{1}{4eg-f^2}\left(-4f\sqrt{e+fx+gx^2} + 2f\sqrt{e+fx+gx^2} + \frac{2fx(f+2gx)}{2\sqrt{e+fx+gx^2}} + \frac{4e(f+2gx)}{2\sqrt{e+fx+gx^2}}\right)$$
$$= \left(\frac{-2f(e+fx+gx^2)+(fx+2e)(f+2gx)}{(4eg-f^2)\sqrt{e+fx+gx^2}}\right) = \left(\frac{(4eg-f^2)x}{(4eg-f^2)\sqrt{e+fx+gx^2}}\right) = \frac{x}{\sqrt{e+fx+gx^2}}.$$

[35] $2\delta = a + \alpha = BM + AK$ is the sum of the minor and major bases of the trapezium, and r is half its height.

[36] For a sphere, Newton employed Case 2, Form 4, of the first catalogue.

> Then the force by which the spheroid attracts the body P will be to the force by which a sphere described with diameter AB attracts the same body as $(SA \times SCq - PS \times KMRK)/(PSq + SCq - SAq)$ to $SAcub./3PSquad.$[37]

12.4 Concluding Remarks

I wrote at length about these two corollaries to present some examples of how quadratures occurring in the *Principia* were discussed by Newton and his acolytes. These are not a reconstruction of what Newton might have been able to do but exactly how he analytically resolved certain problems in the *Principia*. There are, indeed, a number of problems in the *Principia* that are resolved "granting the quadrature of curvilinear figures." Newton tackled these problems by the squaring of curves but gave no details in the printed text about how such quadratures could be performed. Newton's contemporaries were aware of the fact that Newton was hiding the analysis of these problems. As Fontenelle stated,

> Furthermore, it is a justice due to the learned M. Newton, and that M. Leibniz himself accorded to him: That he has also found something similar to the differential calculus, as it appears in his excellent book entitled *Philosophiae naturalis principia mathematica*, published in 1687, which is almost entirely about this calculus.[38]

In some cases, as with Corollary 3, Proposition 41, Book 1, and Corollary 2, Proposition 91, Book 1, it is possible to recover Newton's analysis on its own terms.

Newton's fluxional analysis is not to be conflated either with Leibniz's calculus or with later developments achieved by mathematicians such as Euler or Lagrange. It would be foolish to think that Newton wrote the *Principia* in symbolic terms, using perhaps differential equations and starting his work from $F = ma$, and that only in retrospect did he translate everything into geometry; he would have been a kind of crypto-Eulerian *ante-litteram*. The use of fluxional analysis in the *Principia* is not systematic but sporadic and serves the purpose of overcoming difficulties emerging in some isolated specific passages in a demonstrative structure that is essentially geometrical.

Further, it must be stressed that many propositions of the *Principia* are deeply geometrical in character (e.g., Proposition 1, Book 1); Newton would not have

---

[37] *Principles*, pp. 616–7.

[38] "C'est encore une justice dûë au sçavant M. Newton, & que M. Leibniz lui a renduë lui-même: Qu'il avoit aussi trouvé quelque chose de semblable au calcul différentiel, comme il paroît par l'excellent Livre intitulé *Philosophiae Naturalis Principia Mathematica*, qu'il nous donna en 1687, lequel est presque tout de ce calcul." L'Hospital, *Analyse des Infiniment Petits* (1696), p. xiv. For Newton's quotation of these lines, see Cohen, *Introduction to Newton's Principia* (1971), p. 294.

known how to translate them into algebraic symbols. The translation of the *Principia* into the language of the calculus was a laborious process that required the efforts of three generations of very skilled mathematicians. With the benefit of hindsight it is always possible to present in symbolic form what Newton did with his characteristic blend of geometry and algorithm. But such translations obfuscate both Newton's style and the momentous stature of the contributions of the mathematicians (very often continental) who set themselves the task of writing analytical mechanics in terms of differential and integral calculus.[39]

There is no hidden algebraic *Principia* to be found in a private collection of lost Newtoniana. Both those who have maintained and those who have denied this myth, which Hall quite appropriately calls the "fable of fluxions,"[40] have misunderstood Newton's pronouncements on the use of analysis in his *magnum opus*. When Newton referred to analysis and synthesis, he was not contrasting algebra and geometry, and this in the first place because he was convinced that there was a geometrical as well as an algebraic analysis. He was rather referring to a canon of problem solving that has to be read in its historical context. Indeed, Newton claimed to be a follower of a canon well established in his time. That is, in some instances he analyzed problems by translating them into algebraic equations (see Proposition 30, Book 1) or fluxional equations (see Corollary 3, Proposition 41, and Corollary 2, Proposition 91, Book 1). As far as algebraic equations, after having resolved them he constructed their solutions geometrically. As he often stated, only constructions are worthy of being published, the equations have to be neglected. As far as fluxional equations are concerned, Newton followed the practice of hiding them; he printed only the geometrical construction of their fluent roots. As he stated in 1670,

> After the area of some curve has thus been found, careful considerations should be given to fabricating a demonstration of the construction which as far as permissible has no algebraic calculation, so that the theorem embellished with it may turn out worthy of public utterance.[41]

Here I finally have to note a failure in Newton's program. He often repeated that only geometrical constructions and their demonstrations are compatible with what the ancient geometers did; the force of geometry and its certainty lay "in its splendidly composed demonstrations."[42] However, many geometrical constructions of the *Principia* only hide the fluxional analysis—most notably Newton's analytical quadrature techniques—without replacing it with a geometrical demonstration that

[39] Blay, *La Naissance de la Mécanique Analytique* (1992).
[40] Hall, *Isaac Newton: Adventurer in Thought* (1992), pp. 212–13.
[41] MP, 3, p. 279. "Postquam Curvae alicujus area sic inventa fuerit; de constructionis demonstratione consulendum est, quacum sine Computo Algebraico quantum liceat contexta ornetur Theorema ut evadat publicae notitiae dignum." MP, 3, p. 278.
[42] See, for instance, MP, 8, pp. 454–5.

can stand on its own independently of the analysis. Consider Newton's geometrical construction of the resolution of Corollary 2, Proposition 91, Book 1. Such a construction is far from being a splendid demonstration; it is simply a geometrical translation (12.32) of the analytical resolution (12.31) of a quadrature problem. The fact is that Newton's new analysis was far more powerful than the geometry of the ancients that he praised so consistently. Some of the results of the *Principia* could be obtained only thanks to the highly symbolical manipulations displayed in his two catalogues of curves of *De Methodis* and the new analysis of the method of series and fluxions, the innovative algorithm Newton devised in his *anni mirabiles*. After the publication of the *Principia*, Newton worried in his private ruminations about this unresolved tension between ancient solutions and modern resolutions (see part V).

V Ancients and Moderns

Part 5 covers a period from the publication of the *Principia* in 1687 to the eruption of the polemic with Leibniz in late 1711. These are the years in which Newton became not only the most famous British natural philosopher but also a public figure when he moved to London as Warden (and then Master) of the Mint and was elected President of the Royal Society. Now Newton had the status and power to enforce his program as a natural philosopher, especially at the Royal Society but also beyond the walls of London by fostering the academic success of his acolytes in the English and Scottish universities. These are years in which Newton intensely rethought his previous work, attempting a restructuring of the *Principia* and bringing to light his long awaited *Opticks* (1704).

In this context Newton as a mathematician wrote the most philosophy-laden texts of his production. He engineered a reconstruction of the history of mathematics (from its ancient heights to its recent corruption) aimed at supporting his mathematical methodology. He did so by stressing continuity between his mathematical practice and the Pappian method of analysis and synthesis, and by highlighting discontinuity with Cartesian (and later with Leibnizian) algorithms. He based this reconstruction mainly on evidence derived from the seventh and eighth books of Pappus's *Collectio*.

This historicist approach had a momentous impact on Newton's conceptions of the role of mathematics in natural philosophy. Chapter 13 details what Newton stated about the relations between mechanics and geometry in his preface to the *Principia* and a series of texts related to it.

Chapter 14 focuses on the geometrical works that Newton wrote in the 1690s, most notably "Geometriae Libri Duo," and on the pronouncements on the mathematical methods in natural philosophy that he publicized in the first Latin edition of the *Opticks* (1706). In this context Newton wrote about the dual method of analysis and synthesis, about the comparison between the ancients and the modern mathematicians, and about the superiority of geometry over algebra. These methodological, idiosyncratic, and verbose writings form the basis of Newton's reaction to Leibniz's calculus, which is studied in part VI.

13 Geometry and Mechanics

> Therefore geometry is founded on mechanical practice and is nothing other than that part of universal mechanics which reduces the art of measuring to exact proportions and demonstrations.
>
> —Isaac Newton, 1687

13.1 The *Preface* to the *Principia*

The first edition of Newton's *Principia* opens with a "Praefatio ad Lectorem."[1] The first lines of this *Preface* have received scant attention from historians, even though they contain the very first words addressed to the reader of one of the greatest classics of science. Instead, it is the second half of the *Preface* to which historians have often referred in connection with their treatments of Newton's scientific methodology.

Roughly in the middle of the *Preface*, Newton defined the purpose of philosophy as a twofold task: to investigate the forces of the phenomena of nature and, having established the forces, to demonstrate the remaining phenomena. Newton then introduced a distinction between the first two books, which deal with general propositions, and the third, where the propositions are applied in particular instances to celestial phenomena. From these phenomena, Newton claimed, the force of gravity, thanks to which bodies tend toward the sun, is derived.[2] By assuming this force by means of mathematical propositions, other motions are deduced: the motions of planets, the comets, the moon, and the sea.[3] Newton then declared his hope that phenomena relative to small particles would also be explained through the understanding of attractive (or repulsive) forces hitherto unknown to philosophers.[4] The *Preface* ends with a *laudatio* of Edmond Halley and an apologetic

---

Epigraph from *Principles*, p. 382.

[1] *Principia*, pp. 15–7 = *Principles*, pp. 381–3. In what follows, I refer to the "Praefatio" as the *Preface*. In fact, in the first edition (1687) of the *Principia*, Halley's ode follows a "Praefatio ad Lectorem." In the second (1713) and third (1726) editions, this became the "Auctoris Praefatio ad Lectorem" following the ode. There are no significant variants in the text of the *Preface*.

[2] On the complex route followed by Newton in the application of the mathematical propositions of the first two books to the celestial phenomena studied in Book 3, see Cohen, *The Newtonian Revolution* (1980) and Smith, "The Methodology of the *Principia*" (2002).

[3] For a good review of Newton's achievements, see Wilson, "Newton and Celestial Mechanics" (2002).

[4] See McMullin, *Newton on Matter and Activity* (1978).

passage concerning certain imperfections in the presentation of advanced subjects, such as the moon's motions. It is these lines to which historians have given most attention. I take them up at the end of the next chapter (§14.3).

The first half of the *Preface* is devoted to defining rational mechanics, in contrast to practical mechanics, and to discussing its relation with geometry and its use in the investigation of nature. Most notably, Newton claimed that geometry is founded upon mechanics, whereas one would expect mechanics to be defined as an application of geometry to the science of motion, as, for instance, in Wallis's *Mechanica* (1670–1671).[5] These lines are somewhat difficult to interpret and are, with a few notable exceptions,[6] ignored in the literature.[7] Why this neglect? Perhaps because, on the one hand, general historians of science often ignore Newton's mathematical studies, in which Newton rooted the language of the *Preface*, and on the other, historians of mathematics cannot find exciting new mathematical discoveries in Newton's opening address to the reader.

In this chapter I interpret the opening lines of the *Preface* as a criticism of Descartes' theories of geometry, mechanics, and certainty as laid out in the *Géométrie*. The *Principia* was written—from its title to the concluding General Scholium—as a criticism of Cartesian ideas, so it is not surprising to find anti-Cartesianism in its opening lines. I also show why such criticism was crucial enough for Newton to be put at the very beginning of his *magnum opus* (§13.3).

I cite at length from the opening lines of the *Preface*:

> *Preface* to the Reader
>
> Since the ancients (according to Pappus), considered mechanics to be of greatest importance in the investigation of nature and science and since the moderns—rejecting

---

[5] "eam Geometriae partem ... quae Motum tractat." Wallis, *Opera*, 1, p. 575.

[6] The only studies devoted to the opening lines of the *Preface* that I am aware of, and to which I am deeply indebted, are Garrison, "Newton and the Relation of Mathematics to Natural Philosophy" (1987); Gabbey, "Newton's Mathematical Principles of Natural Philosophy" (1992); Dear, *Discipline and Experience* (1995), pp. 210ff; and Domski, "The Constructible and the Intelligible in Newton's Philosophy of Geometry" (2003). Garrison and Domski are particularly useful for my purposes because they delve deeply into Newton's mathematical methodology. Domski's paper is an important contribution because, for the first time, Descartes' *Géométrie* is identified as a polemic target motivating Newton's *Preface*.

[7] For example, Cohen, *Guide to Newton's Principia* (in *Principles*, pp. 1–370), in which every section of the *magnum opus* is discussed and elucidated, devotes no space to the *Preface*. Similarly, in *Newton's Principia for the Common Reader* (1995), a detailed mathematical analysis of all the propositions of Books 1 and 3, Chandrasekhar skips the *Preface* altogether. In *The Key to Newton's Dynamics* (1995), a lucid and influential introduction to the grounding propositions of the *Principia*, Brackenridge devotes a passing reference to the *Preface*: "[I]t opens with a reference to the ancients and closes with an appeal to the reader to look with patience to Newton's 'labors in a field so difficult'" (p. 142). In *Newton's Principia, the Central Argument* (1995), Densmore does not comment on the *Preface*. The *Preface* is also ignored in De Gandt, *Force and Geometry in Newton's Principia* (1995).

substantial forms and occult qualities—have undertaken to reduce the phenomena of nature to mathematical laws, it has seemed best in this treatise to concentrate on mathematics as it relates to natural philosophy. The ancients divided mechanics into two parts: the rational, which proceeds rigorously through demonstrations; and the practical. Practical mechanics is the subject that comprises all the manual arts, from which the subject of mechanics as a whole has adopted its name. But since those who practice an art do not generally work with a high degree of exactness, the whole subject of mechanics is distinguished from geometry by the attribution of exactness to geometry; and of anything less than exactness to mechanics. Yet the errors do not come from the art, but from those who practice the art. Anyone who works with less exactness is a more imperfect mechanic; and if anyone could work with greatest exactness, he would be the most perfect mechanic of all. For the description of straight lines and circles, which is the foundation of geometry, appertains to mechanics. Geometry does not teach how to describe these straight lines, but postulates such a description. For geometry postulates that a beginner has learnt to describe lines and circles exactly before he approaches the threshold of geometry, and then it teaches how problems are solved by these operations. To describe straight lines and to describe circles are problems, but not problems in geometry. Geometry postulates the solution of these problems from mechanics; and teaches the use of the problems thus solved. And geometry can boast that with so few principles obtained from other field, it can do so much. Therefore geometry is founded on mechanical practice and is nothing other than that part of universal mechanics which reduces the art of measuring to exact proportions and demonstrations. But since the manual arts are applied especially to making bodies move, geometry is commonly used in reference to magnitude, and mechanics in reference to motion. In this sense rational mechanics will be the science, expressed in exact proportions and demonstrations, of the motions that result from any forces whatever and of the forces that are required for any motions whatever. The ancients studied this part of mechanics in terms of the five powers that relate to the manual arts, and paid hardly any attention to gravity (since it is not a manual power) except in the moving of weights by these powers. But since we are concerned with natural philosophy rather then manual arts and are writing about natural rather than manual powers, we concentrate on aspects of gravity, levity, elastic forces, resistance of fluids, and forces of this sort, whether attractive or impulsive. And therefore our present work sets forth mathematical principles of natural philosophy. For the basic problem [whole difficulty] of philosophy seems to be to discover the forces of nature from the phenomena of motions and then to demonstrate the other phenomena from these forces.[8]

[8] *Principles*, pp. 381–2. I quote the original here with line numbers in square brackets preceding the lines: “Praefatio ad Lectorem [1] Cum veteres mechanicam (uti auctor est Pappus) in rerum [2] naturalium investigatione maximi fecerint; & recentiores, missis [3] formis substantialibus & qualitatibus occultis, phaenomena naturae ad [4] leges mathematicas revocare aggressi sint: Visum est in hoc tractatu [5] mathesin excolere, quatenus ea ad philosophiam spectat. Mechani- [6] cam vero duplicem veteres consituerunt: rationalem, quae per demon- [7] strationes accurate procedit, & practicam. Ad practicam spectant [8] artes omnes manuales, a quibus utique mechanica nomen mutata est. [9] Cum autem artifices parum accurate operari soleant, fit ut mechanica [10] omnis a geometria ita distinguatur, ut quicquid accuratum sit ad ge- [11] ometriam referatur, quicquid minus accuratum ad mechanicam. At- [12] tamen errores non sunt artis,

Newton began the *Preface* by defining rational mechanics in contrast to practical mechanics. It is not hard to find Newton's source in this work of disciplinary taxonomy, since it is explicitly cited: Pappus. In the opening lines of Book 8 of the *Collectio*, one finds a long laudatory presentation of mechanics in the form of an address to Hermodorus:

> [S]ince the mechanical inquiry, Hermodorus my son, leads to many important questions in our life, it is rightly held by philosophers to be worthy of the highest esteem, and all the mathematicians cultivate it with not indifferent attention, and in fact it is almost the first to deal with physiology, which concerns the matter of the world's elements ... According to Heron the mechanician, in reality one part of mechanics is rational, while the other part needs the manual work.[9]

By paraphrasing Pappus in the first lines of the *Principia* (1–8),[10] Newton located his work within a tradition dating back to those ancients who used mechanics in

---

sed artificum. Qui minus accurate ope- [13] ratur, imperfectior est mechanicus, & si quis accuratissime operari [14] posset, hic foret mechanicus omnium perfectissimus. Nam & linearum [15] rectarum & circulorum descriptiones, in quibus geometria fundatur, [16] ad mechanicam pertinent. Has lineas describere geometria non docet, [17] sed postulat. Postulat enim ut tyro easdem accurate describere prius [18] didiceret, quam limen attingat geometriae; dein, quomodo per has ope- [19] rationes problemata solvantur, docet; rectas & circulos describere [20] problemata sunt, sed non geometrica. Ex mechanica postulatur horum [21] solutio, in geometria docetur solutorum usus. Ac gloriatur geome- [22] tria quod tam paucis principiis aliunde petitis tam multa praestet. [23] Fundatur igitur geometria in praxi mechanica, & nihil aliud est quam [24] mechanicae universalis pars illa, quae artem mensurandi accurate pro- [25] ponit ac demonstrat. Cum autem artes manuales in corporibus moven- [26] dis praecipue versentur, fit ut geometria ad magnitudinem, mechani- [27] ca ad motum vulgo referatur. Quo sensu mechanica rationalis erit [28] scientia motuum, qui ex viribus quibuscunque resultant, & virium [29] quae ad motus quoscunque requiruntur, accurate proposita ac demon- [30] strata. Pars haec mechanicae a veteribus in potentiis quinque ad ar- [31] tes manuales spectantibus exculta fuit, qui gravitatem (cum potentia [32] manualis non sit) vix aliter quam in ponderibus per potentias illas [33] movendis considerarunt. Nos autem non artibus sed philosophiae con- [34] sulentes, deque potentiis non manualibus sed naturalibus scribentes, ea [35] maxime tractamus, quae ad gravitatem, levitatem, vim elasticam, re- [36] sistentiam fluidorum & ejusmodi vires eu attractivas seu impulsivas [37] spectant: Et ea propter, haec nostra tanquam philosophiae *Principia* [38] mathematica proponimus. Omnis enim philosophiae difficultas in eo [39] versari videtur, ut a phaenomenis motuum investigemus vires naturae, [40] deinde ab his viribus demonstremus phaenomena reliqua." *Principia*, pp. 15–6.

[9] Commandino's Latin: "Cum mechanica contemplatio fili Hermodore multis, & magnis vitae nostrae rationibus conducat, iure optimo a philosophis maxima laude digna existimata est: & omnes mathematici non mediocri studio in eam incumbunt; etenim fere prima physiologiam, quae in elementorum mundi materia versatur, attingit ... Mechanicae vero alteram partem rationalem esse, alteram manuum opera indigere, sentit Hero mechanicus." Pappus, *Mathematicae Collectiones* (1588), p. 305. See Cuomo, *Pappus of Alexandria and the Mathematics of Late Antiquity* (2000), p. 91, for an English translation from the Greek.

[10] In what follows, I use line numbers in parentheses referring to the Latin *Preface* as printed in the third edition (1726) of the *Principia*, available in facsimile reproduction in Newton, *Principia*, pp. 15–6 (see footnote 8).

the investigation of nature. The moderns were also cited, with approbation, since they refused Aristotelian substantial forms and occult qualities in order to reduce natural phenomena to mathematical laws (2–4).

According to Newton, Pappus taught that the ancients further distinguished between rational and practical mechanics (5–7). The latter did not concern Newton because it deals with manual arts, *machinae* constructed by humankind for practical purposes. Newton was concerned, rather, with *philosophia naturalis*, with the mathematical study of natural phenomena (5). The term mechanics, Newton noted, is often used as shorthand for practical mechanics, the art of mechanics applied for practical purposes (7–8). He then proceeded in the next two lines (9–11) to say that "since those who practice an art [*artifices*] do not generally work with a high degree of exactness [*accurate*], the whole subject of *mechanics* is distinguished from *geometry* by the attribution of exactness to *geometry*; and of anything less than exactness to *mechanics*." In the following lines (11–14), Newton sharply criticized the characterization of mechanics as lacking in the exactness proper to geometry. The errors of practical mechanics are caused by "those who practice the art [*artifices*]." In the application of mechanics to the construction of machines people necessarily realize imperfect constructs. On the other hand, the workings of nature are regulated by mathematical laws studied by rational mechanics. Here, no error occurs and the greatest exactness reigns, since the world is under the rule of a *mechanicus omnium perfectissimus* (14).[11]

Rather than excluding mechanics from the realm of geometrical exactness, Newton proposed to subsume geometry under mechanics. Geometry is founded upon mechanics, since the description (or the construction) of geometrical objects appertains to mechanics. For instance, geometry reasons about straight lines and circles, but the description of straight lines and circles is the business of mechanics (14–16). Geometry does not teach how to describe these lines but rather postulates that such a description is accomplished (16–17). What are generally called problems of geometry (i.e., "to describe such and such a figure") are really, according to Newton, problems of mechanics (17–21):

> [F]or geometry postulates [*postulat*] that a beginner has learnt to describe lines and circles exactly [*accurate*] before he approaches the threshold of geometry; and then it teaches how problems are solved by these operations. ... To describe straight lines and to describe circles are problems, but not problems in geometry. Geometry postulates the solution of these problems from mechanics; and teaches the use of the problems thus solved.[12]

[11] One might note that in the *Praefatio Authoris* of Copernicus' *De Revolutionibus* (1543), p. 3 the world was described as created by "the best and most systematic Artisan of all" = "ab optimo et rugularis, omnium Opifice." One might thus surmise that here Newton is referring to a wise and providential God.

[12] Newton, *Principles*, p. 382.

Geometrical figures are generated by motion, and the solution of geometrical problems is possible if one knows how the figures are kinematically generated. Rational mechanics is thus not only as exact as geometry but precedes geometry, since it generates the geometrical objects. Mamiani noted[13]—I believe correctly—a touch of irony in the following sentence: "[a]nd geometry can boast that with so few principles obtained from other fields, it can do so much."[14] Indeed, the rhetoric of self-sufficiency that pervades so many writings about geometry had no place in Newton's conception because geometry is nothing other than a part of universal mechanics: "[G]eometry is founded on mechanical practice and is nothing other than that part of universal mechanics which reduces the art of measuring to exact proportions and demonstrations" (23–25).[15]

I was unable to locate other places where Newton employed the term mechanica universalis. This term has some resemblance to "Arithmetica Universalis," the title of one of Newton's major mathematical works. The reason this mathematical discipline (now called algebra) is defined as universal is that it deals both with practical arithmetic (the science of "numbers and numerable things")[16] and the abstract manipulation of letters standing for constants and variables (this discipline was often called arithmetica speciosa).[17] It is tempting to interpret "universal mechanics" as a discipline that comprises both rational and practical mechanics. However, the idea conveyed by the lines I have quoted is clear: geometrical objects are generated by motion, and that is why their description falls within the scope of mechanics.

The fact that geometry is founded upon mechanical practice yields, according to Newton, another important consequence. Geometry can be applied to natural philosophy, that is, to the mathematical study of force and motion (25–30):

> [B]ut since the manual arts are applied especially to making bodies move, geometry is commonly used in reference to magnitude, and mechanics in reference to motion. In this sense rational mechanics will be the science, expressed in exact proportions and demonstrations, of the motions that result from any forces whatever and of the forces that are required for any motions whatever.[18]

The message is that rational mechanics does not lack in exactness. It is a science of motion whose demonstrations are accurate (29), and it is the foundation of geometry (23). As readers of the *Principia* know, the double task of deducing forces from

[13] Mamiani, "La Rivoluzione Incompiuta" (1998), p. 39.
[14] Newton, *Principles*, p. 382.
[15] Ibid. "that accurately sets forth and demonstrates the art of measuring" in Densmore, *Newton's Principia, the Central Argument* (1995), p. 1.
[16] MP, 8, p. 173.
[17] "Nuperi veterum inventis addere studentes, Arithmeticam speciosam conjunxerunt cum Geometria." MP, 4, p. 420.
[18] Newton, *Principles*, p. 382. I have slightly altered Cohen and Whitman's translation.

motions, and motions from forces (28–29), is the essence of Newton's three books and indeed the essence of analytical mechanics to the present day. Nowadays, the tools of the differential and integral calculus are used to tackle this double task. This is different, Newton stated, from what the ancients did, since they concerned themselves with the "five powers that relate to the manual arts" (30–33).[19] Newton, of course, was concerned with natural powers: attractive or impulsive forces occurring in nature such as gravity, levity, elasticity, and fluid resistance (33–38).

From what has been said, it should be clear that Newton, by subsuming geometry under mechanics and by defending the exactness of mechanics, declared himself a follower of a mathematical methodology that contrasts sharply with Descartes' exclusion of mechanical curves from the realm of the exactitude and certainty of geometry (see chapter 3). My reading of the *Preface* as a criticism of Cartesian mathematical method is reinforced by the concordances with some manuscripts that Newton wrote during the composition and after the publication of the *Principia*, namely, an unpublished draft of the Scholium to Proposition 22, Book 1; the opening of a multipartite treatise on geometry (mid-1690s); and an intended revised *Preface* to the *Principia* (late 1710s), where several of the themes cryptically exposed in the published *Preface* are spelled out.

13.2 Mathematical Manuscripts Related to the *Preface*

The first text of interest comes from a draft of the *Principia*, in the hand of Newton's amanuensis, which was deposited by Newton in the University Library of Cambridge in accomplishment of his duties as Lucasian Professor.[20] In this manuscript, the Scholium to Proposition 22, Book 1, contains an explicit reference to the *Preface*. In the published Scholium these lines were suppressed. Proposition 22 is one of the basic results of Section 5, Book 1, devoted to the geometry of conics. In this proposition Newton gave two methods for organically describing a conic passing through five given points (§5.4.4). This proposition illustrates how geometrical objects are generated mechanically, and indeed in the suppressed Scholium, Newton referred back to what he had stated in the *Preface*:

> [B]y this method points on the trajectory are most readily found, unless you prefer, as in the second case, to describe the curve mechanically. [For the description of curves by motion is the province of mechanics. Geometry does not teach how to describe the straight line and circle, but postulates them as drawn; that is, it postulates that before ever a beginner starts to be a geometer he shall have learned their descriptions].[21]

[19] The five powers being the simple machines of ancient mechanics.

[20] Dd.9.46, fol. 44 (Cambridge University Library). Reproduced in *The Preliminary Manuscripts for Isaac Newton's 1687 Principia, 1684–1686* (1989), p. 135, and Newton, MP, 6, p. 260.

[21] MP, 6, p. 261. "Hac methodo puncta trajectoriae inveniuntur expeditissime, nisi mavis curvam

The mechanical description of conics presented in Proposition 22 is just one example of how other geometrical objects can be generated mechanically. Recall that in Newton's *Arithmetica Universalis* the problem tackled in Proposition 22 is also solved in terms of Cartesian algebra (§5.4.7).[22] But in the *Principia*, Newton preferred to present a mechanical solution, rather than an algebraic one.

In the 1690s, Newton attempted to write a long treatise on geometry ("Geometriae Libri Duo") (see chapter 14). In some passages in the *incipit* of a version of this geometrical work Newton quoted verbatim from the *Preface*. This reveals that the *Preface* to the *Principia* must be read as part of Newton's reflections on the scope and methods of geometry. Indeed, in defining the scope of geometry, Newton immediately warned the reader that the "genesis of the subject-matter of geometry" appertains to mechanics, and he did so in terms very similar to the *Preface*:

> [G]eometry neither teaches how to describe a plane nor postulates its description, though this is its whole foundation. To be sure, the planes of fields are not formed by the practitioner [*ab artifice*] but merely measured. Geometry does not teach how to describe a straight line and a circle but postulates them; in other words, it postulates that the practitioner has learnt these operations before he attains the threshold of geometry. . . . Both the genesis of the subject-matter of geometry, therefore, and the fabrication of its postulates pertain to mechanics. Any plane figure executed by God, nature or any technician [*a Deo Natura Artifice quovis confectas*] you will are measured by geometry by the hypothesis that they are exactly constructed.[23]

Note that Newton considered God, nature, and technicians as *artifices* who can generate geometrical objects. This reinforces my tentative interpretation of the *mechanicus omnium perfectissimus*, referred to in the *Preface*, as God. I expand on this theme in the next chapter (§14.2).

Newton continued the opening of "Geometriae Libri Duo" with critical remarks addressed to Cartesian mathematical methodology. Namely, he criticized Descartes'

---

ut in casu secundo describere Mechanice. [Nam curvarum descriptio per motum ad Mechanicam pertinet. Rectam et circulum describere Geometria non docet sed postulat, id est postulat Tyronem antequam is incipit esse Geometra descriptiones eorum didicisse]." MP, 6, p. 260. I have put in square brackets passages that were suppressed in the printed *Principia*.

[22] Newton chose the coordinate axes so that the general equation of the conic has the form $a + bx + cx^2 + dy + exy + y^2 = 0$. The five conditions of the problem (i.e., the fact that the conic must pass through five given points) translate into a system of five equations, which, when solved, determine the coefficients. MP, 5, pp. 305–15.

[23] MP, 7, pp. 287, 289. "Planum describere Geometria nec docet nec postulat quamvis hoc sit Geometriae totius fundamentum. Quippe plana agrorum non formantur ab artifice sed mensurantur tantum. Lineam rectam et circulum describere Geometria non docet sed postulat, hoc est postulat Artificem has operationes prius didicisse quam attingit limen Geometriae. . . . Pertinet igitur ad Mechanicam tum genesis subjecti Geometrici tum Postulatorum effectio. Figuras quasvis planas a Deo Natura Artifice quovis confectas Geometra ex hypothesi quod sunt exacte fabricatae mensurat." MP, 7, pp. 286, 288.

exclusion of mechanical curves from geometry and the definition of mechanics as lacking exactness:

> [A] technician is required and postulated to have learnt how to describe straight lines and circles before he may begin to be a geometer. And it consequently does not matter how [a]by what mechanical means[a] they shall be described. Geometry does not posit modes of description: we are free to describe them [plane figures] by moving rulers around, using optical rays, taut threads, compasses, the angle given in a circumference, points separately ascertained, the unfettered motion of a careful hand, or finally any mechanical means whatever. Geometry makes the unique demand that they are described exactly.[24]

Any motion executed by the *artifex* generates a figure that can be taken as a legitimate object of geometry, provided that the *artifex* generates it with exactness. According to Newton, the various mechanical generations of curves listed in the preceding passage are legitimate (*contra* Descartes). This makes, for instance, spirals perfectly legitimate curves, provided that the mechanical means that generate them are operated with exactness. God and nature (but not the technicians) are certainly able to operate with the required exactness. According to Newton, this was also well known to the ancients, who accepted mechanical curves as legitimate. Indeed—as one reads in the opening lines of Book 2 of "Geometriae Libri Duo"—according to the ancients, the straight line, the circle, and the conics are also mechanical, since their generation is conceived in terms of ruler, compass, and cone sections. Further, the ancients "as Pappus recounts, assuredly did not shrink from admitting further curves."[25]

According to Newton, it is a mistake to attribute to mechanics the imperfection that is proper to the manual arts exercised by human technicians:

> [I]t has now [by the Cartesians?], however, come to be usual to regard as geometrical everything which is exact, and as mechanical all that proves not to be of the kind, as though nothing could possibly be mechanical and at same time exact. But this common belief is a stupid [*crassa vulgi opinio*] one, and has its origin in nothing else than that geometry postulates an exact mechanical practice in the description of a straight line and a circle, and moreover is exact in all its operations, while mechanics as it is commonly exercised is imperfect and without exact laws. It is from the ignorance and imperfection of mechanicians that the common opinion defines mechanics.[26]

---

[24] Ibid. [aa] canceled. "Rectas et circulos describere Artifex prius didicisse requiritur et postulatur quam incipit esse Geometra. Ideo nil refert quomodo [a]qua ratione mechanica[a] describantur. Geometra modos descriptionum non ponit. Regulis admotis, radijs opticis, funiculis tensis, circino, angulo dato in perimetro, punctis discretim inventis, manus exquisitae motu libero, ratione denique quacunque mechanica eas describere permittimur. Id solum postulat Geometria ut describantur exacte."

[25] MP, 7, p. 385.

[26] MP, 7, p. 289. "Usu tamen jam venit Geometricum censere id omne quod exactum est et

Mechanics thus provides geometry with its subject matter, and it does so with a rich variety of mechanical constructions. Echoing the *Preface* to the *Principia*, Newton stated that the glory of geometry is "that it should with so few things solved and granted from outside it by itself perform so many."[27]

A few lines later, Newton added an important observation:

> [I]n definitions it is allowable to posit the reason for a mechanical genesis, in that the species of magnitude is best understood from the reason for its genesis.[28]

When one defines geometrical magnitudes mechanically, one understands the reason for their genesis. Here is an echo of what had been debated since its inception with Alessandro Piccolomini of the *quaestio de certitudine mathematicarum* (1547), and had been discussed in England by Hobbes and Barrow, who maintained that a mechanically based geometry is a discipline endowed with scientific character insofar as it yields knowledge of causes. According to Newton, a mechanically based geometry achieves exactly this end.[29] Newton's downgrading of Cartesian algebra and the Leibnizian calculus to mere heuristic tools devoid of scientific character is thus based on his adoption of the idea that the symbolism of algebra and calculus do not capture the reasons for the genesis of figures (§14.3.3).

A revised version of the opening lines of Book 1 of "Geometriae Libri Duo" is even more closely related to the *Preface* to the *Principia*.[30] It begins with a reference to Book 8 of Pappus's *Collectio*, from which Newton derived the now familiar distinction between rational, speculative and accurate mechanics and manual and not accurate mechanics. This version continues by paraphrasing the *Preface* and ends in defiance against the modern geometers (namely, the Cartesian mathematicians):

> But if the authority of the modern geometers is raised against us, even greater is the authority of the ancients.[31]

---

mechanicum quod ejusmodi non existit, quasi nihil mechanicum et simul exactum esse posset. Crassa vero est haec vulgi opinio et non aliunde orta quam quod Geometria postulat exactam praxin mechanicam in descriptione rectae et circluli, et praeterea in omnibus suis operationibus exacta est, Mechanica vero imperfecte et absque legibus exactis vulgo exercetur. Ex imperitia et imperfectione Mechanicorum Vulgus definit Mechanicam." MP, 7, p. 288. Cf. van Schooten's annotation in Descartes, *Geometria*, p. 18.

[27] MP, 7, p. 291. "Geometriae ... gloria quod tam paucis aliunde solutis et concessis tam multa suo marte praestet." MP, 7, p. 290.

[28] MP, 7, p. 291. "In definitionibus ponere licet rationem geneseos Mechanicae, eo quod species magnitudinum ex ratione geneseos optime intelligitur." MP, 7, p. 290.

[29] Piccolomini, *In Mechanicas Quaestiones Aristotelis* (1547). On the Quaestio, see Mancosu, *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century* (1996).

[30] MP, 7, pp. 338–43.

[31] "Et si authoritas novorum Geometrarum contra nos facit, tamen major est authoritas Veterum." MP, 7, p. 342.

Newton's predilection for the ancient geometers also emerges explicitly in an intended *Preface* to the *Principia* (hereafter, the *Intended Preface*). In the late 1710s, Newton wrote a new *Preface* to the *Principia* that would have replaced the one published in 1687.[32] This project did not materialize, and the *Preface* remained unchanged through its second and third editions. However, the *Intended Preface* sheds light on the published one. Indeed, Newton planned to unfold several themes alluded to in the published *Principia* in a more explicit way. The *Intended Preface* would have begun with an explicit praise for the geometry of the ancients:

> [T]he ancient geometers investigated things sought through analysis, demonstrated them when found out through synthesis, and published them when demonstrated so that they might be received into geometry. Once analyzed they were not straightaway received into geometry: there was need of their solution through composition of their demonstrations. For the force of geometry and its every merit laid in the utter certainty of its matters, and that certainty in its splendidly composed demonstrations. In this science regard must be paid not only to the conciseness of writing but also to the certainty of things. And on that account I in the following treatise synthetically demonstrated the propositions found out through analysis.[33]

Here Newton justified the geometrical style of the *Principia*, stating that he wanted to adhere to the ancients' way of presenting their theorems by synthesis. Indeed, he shared with many of his contemporaries (Descartes included) the idea that the Greek geometers had possessed an analytical method of discovery that they had kept hidden, for the ancients would rather have published their results according to a synthetic method. Finally, Newton claimed that the use of the analytical method of fluxions was not explicit in the *Principia* because of his desire to adhere to this ancient practice of publication.

Newton next came to a question of possible discontinuity between his geometry and classic geometry, a discontinuity that from the point of view of a twenty-first century historian is apparent. Newton's geometry in the *Principia* appears nowadays as extremely innovative because it is applied to motion and force, velocity and

---

[32] Whiteside has transcribed the Latin text and translated the opening paragraphs of the *Intended Preface* (Add. 3968.9, f. 109) and of three preliminary drafts in private possession (but now, fortunately, in the Macclesfield Collection recently acquired by the University Library of Cambridge) in MP, 8, pp. 442–59; Cohen and Whitman provide an English translation in Newton, *Principles*, p. 49–54. I am deeply indebted to Whiteside, and Cohen and Whitman for their extensive commentaries.

[33] MP, 8, pp. 453, 455. "Geometrae Veteres quaesita investigabant per Analysin, inventa demonstrabant per Synthesin, demonstrata edebant ut in Geometriam reciperentur. Resoluta non statim recipiebantur in Geometriam: opus erat solutione per compositionem demonstrationum. Nam Geometriae vis et laus omnis in certitudine rerum, certitudo in demonstrationibus luculenter compositis constabat. In hac scientia non tam brevitati scribendi quam certitudini rerum consulendum est. Ideoque in sequenti Tractatu Propositiones per Analysin inventas demonstravi synthetice." MP, 8, pp. 452, 454.

acceleration. It is viewed as a mathematical practice deeply related to seventeenth-century kinematic methods, such as those of Roberval and Barrow. In a passage that recalls the *incipit* of *De Quadratura* rather than stressing this element of modernity, Newton reinforces the argument for continuity:

> [T]he geometry of the ancients had, of course, primarily to do with magnitudes, but propositions on magnitudes were from time to time demonstrated by means of local motion: as, for instance, when the equality of triangles in Proposition 4 of Book 1 of Euclid's Elements were demonstrated by transporting either one of the triangles into the other's place. Also, the genesis of magnitudes through continuous motion was received in geometry: when, for instance, a straight line were drawn into a straight line so as to generate an area, and an area were drawn into a straight line to generate a solid. If the straight line which is drawn into another be of given length, there will be generated a parallelogram area. If its length be continuously changed according to some fixed law, a curvilinear area will be generated. If the size of the area drawn into the straight line be continuously changed, there will be generated a solid terminated by a curved surface. If times, forces, motions and speeds of motion be expressed by means of lines, areas, solids or angles, then these quantities too can be treated in geometry. Quantities increasing by continuous flow we call fluents, the speeds of flowing we call fluxions and the momentary increments we call moments, and the method whereby we treat quantities of this sort we call the method of fluxions and moments: this method is either synthetic or analytical.[34]

According to Newton, the ancients, too, conceived geometrical objects as generated by motion, subsuming geometry under mechanics. The practice of demonstrating propositions by means of local motion is thus not against ancient practice. This is the message conveyed in these lines from the *Intended Preface*.

In the 1690s and early 1710s, Newton deepened his interest in the double method of analysis and synthesis. His search for an ancient geometrical analysis, preferable for its conciseness and elegance to the Cartesian algebraic one, and his conviction that only synthesis provides certainty and thus can be uttered publicly pervade

[34] MP, 8, p. 455. "Geometria Veterum versabatur quidem circa magnitudines; sed Propositiones de magnitudinibus nonnunquam demonstrabantur mediante motu locali: ut cum triangulorum aequalitas in Propositione quarta libri primi Elementorum Euclidis demonstraretur transferendo triangulum alterutrum in locum alterius. Sed et genesis magnitudinum per motum continuum recepta fuit in Geometria: ut cum linea recta duceretur in lineam rectam ad generandam aream, & area duceretur in lineam rectam ad generandum solidum. Si recta quae in aliam ducitur datae sit longitudinis generabitur area parallelogramma. Si longitudo ejus lege aliqua certa continuo mutetur, generabitur area curvilinea. Si magnitudo areae in rectam ductae continuo mutetur generabitur solidum superficie curva terminatum. Si tempora, vires, motus et velocitates motuum exponantur per lineas areas solida vel angulos, tractari etiam possunt hae quantitates in Geometria. Quantitates continuo fluxu crescentes vocamus fluentes & velocitates crescendi vocamus fluxiones, & incrementa momentanea vocamus momenta, et methodum qua tractamus ejusmodi quantitates vocamus methodum fluxionum et momentorm: estque haec methodus vel synthetica vel analytica." MP, 8, p. 454.

his writings. He came also to the conclusion, somewhat cryptically hinted at in the second half of the *Preface*, that the method of analysis and synthesis provides the correct demonstrative pattern to be followed in the mathematization of natural philosophy.

In concluding this chapter, I consider a technical and pressing reason that must have led Newton to accept mechanical curves as necessary tools in the mathematization of celestial mechanics.

13.3 Mechanical Curves in the Study of Planetary Motion

Johannes Kepler found that planets move in ellipses having the sun placed at one focus. He also discovered that each planet moves in such a way that the radius vector joining it to the sun sweeps equal areas in equal times. When the elliptic orbit is known, the position of the planet in function of time can thus be found by calculating the area of the focal sector.

Newton devoted Section 6, Book 1, of the *Principia*, entitled "De Inventione Motuum in Orbibus Datis," to the solution of the so-called Kepler problem. The problem consisted in finding the area of a focal sector of the ellipse and was equivalent to the solution for x of the equation $x - e\sin x = z$ (given e and z). When Newton wrote the *Principia*, only approximate solutions were known. Could there be a finite solution? In Lemma 28, Section 6, Newton proved that the answer is negative:

> No oval figure exists whose area, cut off by straight lines at will, can in general be found by means of equations finite in the number of their terms and dimensions.[35]

This lemma is quite general. Given a plane oval curve and a point P inside it, one cuts a sector S via a straight line passing through P. The lemma states that the sector S is not generally expressible by means of a finite algebraic equation in x and y. Pesic has paraphrased Newton's elegant demonstration:

> [P]ick any point inside the oval and let it be the pole about which a line revolves with uniform angular speed. On that line, let a point move away from the pole with speed proportionate to the square of the distance along the line between the pole and the line's intersection with the oval. Then that moving point on the moving line will move in a gyrating spiral, its distance from the pole recording the area swept out by the line. The area of the oval is given by the distance moved by the point over one complete revolution of the line. But as the line continues to sweep over the oval area again and again, the spiral will continue uncoiling to infinity [figure 13.1)]. Hence, it will intersect any straight line drawn across it an infinite number of times, which shows that the degree of the equation of the spiral

[35] *Principles*, p. 511 = "Nulla extat figura ovalis cujus area, rectis pro lubitu abscissa, possit per aequationes numero terminorum ac dimensionum finitas generaliter inveniri." *Principia*, p. 188.

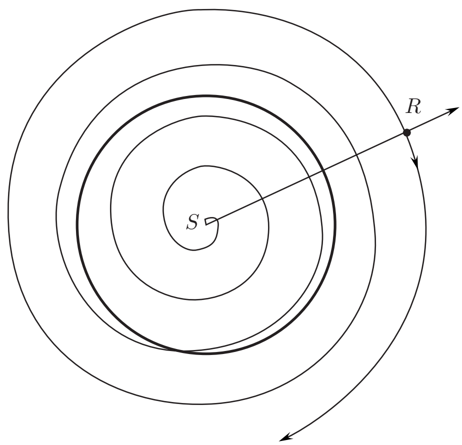


Figure 13.1

Diagram for Lemma 28, Book 1, of Newton's *Principia* for the simplest case, in which the "oval" is a circle and point S is at the circle's center. The line rotates clockwise. Point R moves along the rotating line so that the length SR is proportional to the area of the circle's surface swept by the rotating line. In this case, point R traces an Archimedean spiral. When point S is not at the circle's center or the "oval" is not a circle (e.g., is an ellipse), the spiral traced by R will not have the symmetry that characterizes the Archimedean spiral. Source: By Compomat, s.r.l. ©Niccolò Guicciardini.

> is not finite, since an equation of finite degree can only intersect a given line a finite number of times. Therefore, since the area is given by the equation of the spiral, the area of the curve is not given by an equation of finite degree.[36]

Lemma 28 creates problems of interpretation, since it is unclear what Newton meant by "an oval figure."[37] However, one can certainly assume that the ellipse is an oval figure and that the Lemma applies to the focal sector of the ellipse swept by the radius vector according to Kepler's first two laws. As Newton stated in the Corollary to Lemma 28,

> [H]ence the area of an ellipse that is described by a radius drawn from a focus to a moving body cannot be found, from a time that has been given, by means of a finite equation, and therefore cannot be determined by describing geometrically rational curves. I call curves 'geometrically rational' when all their points can be determined by lengths defined by equations, that is, by involved ratios of lengths, and I call the other curves (such as spirals, quadratrices, and cycloids) "geometrically irrational." For lengths that are or are not as integer to integer (as in Book 10 of the *Elements*)

[36] Pesic, "The Validity of Newton's Lemma 28" ((2001), p. 215.
[37] Pourciau, "The Integrability of Ovals" (2001).

> are arithmetically rational or irrational. Therefore I cut off an area of an ellipse proportional to the time by a geometrically irrational curve as follows.[38]

Note that geometrically rational curves correspond to Descartes' geometrical curves, whereas geometrically irrational curves correspond to Descartes' mechanical ones. The area of the focal sector of the ellipse is measured by the distance SR. The equation of the focal sector cannot be an algebraic function, since for a given angle it should be infinitely valued. Indeed, the spiral is a mechanical curve, not admitted by Descartes' *Géométrie*. Thus, the Kepler problem can be solved only via infinite series (i.e., equations with an infinite number of terms). And infinite series are Newton's main tool to deal with mechanical curves (chapter 7).

In Proposition 31 and its Scholium, Newton showed how one could obtain numerical approximations of the equation $x - e \sin x = z$ thanks to an iterative procedure related to the Newton-Raphson method (§7.5, figure 7.10).[39]

It should be added that Newton was quick to realize that Keplerian planetary theory is just an approximation (albeit a good one for primary planets), since universal gravitation implies perturbations that destroy the simplicity of elliptical orbits. The complexity of the Newtonian cosmological picture opened a gulf (though not that devastating as in the case of Cartesian cosmology) between causal and exact mathematical descriptions of nature. In any case, for primary planets ellipses are a satisfactory approximation (in Book 3 the quiescence of the planets' aphelia was invoked by Newton as the best evidence for inverse-square sun-centered attraction), and thus the Kepler problem remained for Newton, and still remains today, of utmost importance for the astronomer. Indeed, Newton had dealt with this problem since his youthful *anni mirabiles* and had then sketched a proof of the algebraic nonintegrability of ellipses modeled along the lines of Lemma 28.[40] It is, of course, important to keep in mind that Newton, like many of his contemporaries, was interested in dealing with mechanical curves for a broader spectrum of reasons. However, the growing importance of mechanical curves in the study of natural philosophy was an important stimulus for him as well as a further motivation for rejecting Cartesian methodology.

---

[38] *Principles*, p. 513. "Hinc area ellipseos, quae radio ab umbilico ad corpus mobile ducto describitur, non prodit ex dato tempore, per aequationem finitam, & propterea per descriptionem curvarum geometrice rationalium determinari nequit. Curvas geometrice rationales appello quarum puncta omnia per longitudines aequationibus definitas, id est, per longitudinum rationes complicatas, determinari possunt; caeterasque (ut spirales, quadratrices, trochoides) geometrice irrationales. Nam longitudines quae sunt vel non sunt ut numerus ad numerum (quemadmodum in decimo elementorum) sunt arithmetice rationales vel irrationales. Aream igitur ellipseos tempori proportionalem abscindo per curvam geometrice irrationalem ut sequitur." *Principia*, pp. 190–1.

[39] See Adams, "On Newton's Solution of Kepler's Problem" (1882), and Kollerstrom, "Thomas Simpson and Newton's Method of Approximation" (1992).

[40] Around 1665, Newton had already sketched a proof, modeled along the lines of Lemma 28, that the rectification of the ellipse's arc is not algebraic. See Add. 3958.2, f. 34r in MP, 1, p. 545.

Summing up, in Section 6, Book 1, of the *Principia* Newton showed that it was essential for his mathematical cosmology to reject Descartes' conceptions concerning the relations between certainty, mechanics, and geometry, as he made clear in the *Preface.* The *Preface* is thus a complex and stratified text located at the confluence of a number of exigencies that motivated Newton's mathematical natural philosophy. Newton's quest for certainty in natural philosophy, his idea that probabilism and hypotheticism—"that are being blazoned about everywhere," as he complained in his youth—could be avoided by injecting mathematics into cosmology and optics, made it a priority to safeguard the certainty of his mathematical methods. In the *Géométrie*, Descartes had aimed at distinct and clear method by selecting only a well-defined class of constructions that trace curves as loci of polynomial equations. In order to mathematicize Keplerian planetary cosmology, Newton had to surpass such limitations; his concern in the *Preface* was to show that he was not losing certain mathematical ground with such a bold move.

Newton's position is ultimately contradictory. On the one hand, he wished to include mechanical curves as legitimate objects of study and exact means of construction. On the other hand, in order to study mechanical curves he had to resort to techniques in new analysis, as infinite series, which were grounded on algorithmic manipulations based on analogies and Wallisian inductions (§7.1). Such tensions between new analysis (based on computations) and ancient analysis (based on porismatic mechanical tools), and between heuristic analysis and demonstrative synthesis, were broached by Newton in the 1690s and 1710s in fascinating writings on geometry (see chapter 14).

14 Analysis and Synthesis

> But if the authority of the modern geometers is raised against us, even greater is the authority of the ancients.
>
> —Isaac Newton, mid-1690s

14.1 Analysis and Synthesis in "Geometriae Libri Duo"

In the 1690s, Newton researched extensively on geometry. He pursued projects of restoration of the analysis of the ancients that had already occupied him in the 1670s in his work on the Pappus problem. As we mentioned in chapter 5, Newton became convinced that the ancients' method of analysis was related to the organic construction of curves and to the study of projective transmutations between curves. In fact, in his manuscripts on geometrical method he often moved from a Pappian characterization of the method of analysis to a treatment of projective transformations.[1]

Newton's writings on geometry in the 1690s culminated in a two-book treatise, "Geometriae Libri Duo."[2] In the first book he presented his results on projective geometry, classification of cubics, organic description of curves, and geometrical fluxions in the style of "Geometria Curvilinea" (§9.2). The second book is devoted to theorems on quadratures. The first book therefore covers methods (organic, projective, and fluxional) of generation and transformation of curves that Newton considered consonant with the ancients' handling of loci (in his opinion, a basic element of porismatic analysis). Previous drafts, written in the early 1690s, were instead devoted to the ancients' method of analysis, to a comparison of this with

---

Epigraph from MP, 7, p. 342. "Et si authoritas novorum Geometrarum contra nos facit, tamen major est authoritas Veterum.".

[1] This connection is particularly evident in MS Add. 3963.15, ff. 180v/179r, reproduced in MP, 7, pp. 212–6, entitled "In Analysi veterum observandae sunt hae regulae" ("In the Ancients' analysis these rules are to be observed"). Here after a characterization of the method of analysis as one in which one "must consider the problem as accomplished and out of unknown quantities viewed as given gather given ones as unknowns" ("considerandum est Problema tanquam confectum et ex quantitatibus ignotis ceu datis colligendae sunt datae ceu ignotae") (MP, 7, pp. 214–5), Newton moved on to rules on projective correspondences between points.

[2] Newton's writings on geometry can be found in MP, 7, pp. 185–561. The final two-book version of "Geometriae Libri Duo" (which Whiteside reconstructed mostly from Add. 3962.1, 3962.3, 3963.2, and 4004) is in MP, 7, pp. 402–561.

the analysis of the moderns, to questions concerning the nature of synthetic composition, and to the nature of porisms. As Whiteside remarked, the value of Newton's 1690s studies on geometry lies more in the light they shed on his mature methodological views on the nature of mathematics than in the results he achieved, since, with the exception of what he did in classifying cubic curves, he for the most part restated what he had obtained some fifteen years before.[3]

A Proemium to a preliminary version of "Geometriae Libri Duo" is particularly interesting. In the opening section, entitled "What Is To Resolve a Problem, and How To Solve It," Newton dealt with the dual method of resolution (*resolutio* or analysis) and solution (*compositio* or synthesis). After quoting freely from Pappus's famous introduction to Book 7 of the *Collectio*, where the dual method is cryptically presented (§3.1), Newton drew a comparison with algebra, as was typical in the seventeenth century:

> What Pappus here describes is the very thing we do when, by assuming the unknown as known and therefrom by an appropriate argument gathering something known as unknown, we reduce a problem to an equation; and then by aid of that equation we in inverse sequence gather from really known what is really unknown. Nor does our algebra seem to differ from their [the Ancients'] analysis except in the mere manner of its expression.[4]

The understanding of algebra as a form of ancient analysis was common in the seventeenth century. For example, Descartes stated similar ideas in *Regulae*.[5] But just below these lines Newton began with his idiosyncratic invectives against the use of algebra. He made it clear that a problem is solved only through composition, and that in composing its solution no space needs to be given to algebraic criteria. Synthetic composition is perfect only when the analysis is forgotten and eliminated from sight.[6] Composition, or synthesis, must be carried out in such a way that no traces of analysis are evident:

---

[3] "[T]hese uneven writings are memorable less, perhaps, for their individual flashes of innovatory brilliance and sustained displays of masterly technical expertise than for the finely textured picture which they collectively project of Newton's mature attitude *vis-à-vis* the 'ancient' geometers and the 'new' breed of Cartesian mathematicians, of whom, by upbringing and working practice, he himself was one for all that he might defer to the superior logical authority of the former." MP, 7, p. 185.

[4] MP, 7, pp. 249, 251. "Quod Pappus hic describit id ipsum est quod nos facimus ubi assumendo incognitum ut cognitum, et inde per debitam argumentationem colligendo aliquod cognitum ut incognitum, problema ad aequationem deducimus: dein ope aequationis illius ex vere cognitis inverso ordine colligimus vere incognitum. Nec differre videtur Algebra nostra ab illorum Analysi nisi in modulo expressionis." MP, 7, pp. 248, 250.

[5] Descartes, *Regulae ad Directionem Ingenii* (1701) (AT, 10, pp. 376–7).

[6] " [The ancients] regarded a problem as resolved when a geometer had in his own view completed its analysis, and as solved once he had without analysis learnt how to compose it. Whence the solution of problems by the construction of an equation would, to the ancients' mind, seem to be excluded from pure geometry, unless perhaps insofar as an algebraist who is less cognizant

> [S]olution is, however, the opposite of resolution in that it may not be had till all trace of resolution be removed from start to finish by means of a full and perfect composition. For example, if a question be answered by the construction of some equation, that question is resolved by the discovery of the equation and composed by its construction, but it is not solved before the construction's enunciation and its complete demonstration is, with the equation now neglected, composed.[7]

More important, according to Newton, the criteria of analysis should not interfere with synthesis. Descartes had erred in confusing the two levels of method by imposing algebraic criteria on the choice of compositional tools. His way of composition (the construction of the equations, or "to denote the root of a proposed equation geometrically") was unnatural and removed from the simple synthesis of the ancients:

> For almost all problems have a natural way of being solved ... whence happens it, I think, that the ancients, whose aim was composition, frequently arrived at simpler conclusions than the moderns, who are more devoted to algebra.[8]

Thus, not only are the moderns proposing spurious synthetic compositions, but they are following algebraic analyses that are often more complicated compared to geometrical ones. In many cases, according to Newton,

> to solve the problem by algebra alone a man's life would ... not be long enough [a]neither Hercules' patience nor Methuselah's years would suffice[a].[9]

Last but not least, Newton often repeated that geometrical analysis is conducive of a more natural synthesis. Algebraic analysis is not only burdensome; its main

of geometry should propose this particular problem: *To denote the root of a proposed equation geometrically*, or insofar as a geometer should gather from the construction of an equation a solution of a kind propoundable and demonstrable without knowledge of the equation." MP, 7, p. 251. "[Veteres] existimantes Problema resolutum esse quando Geometra apud se absolverat Analysin, solutum quando sine Analysi componere didicerat. Unde solutio problematum per constructionem aequationis e Geometria pura, ex veterum sententia, excludenda videtur: nisi forte quatenus Algebraista qui Geometriam minus intelligit proponat hoc ipsum problema, *Radicem propositae aequationis Geometrice designare*; aut quatenus Geometra ex constructione aequationis colligat ejusmodi solutionem quae sine aequationis notitia proponi ac demonstrari potest." MP, 7, p. 250.

[7] MP, 7, p. 307. "solutionem vero ita contrariam esse resolutioni ut ea non prius habeatur quam resolutio omnis a principio ad finem per compositionem plenam et perfectam excludatur. Verbi gratia si quaestioni per constructionem aequationis alicujus respondeatur, quaestio illa resolvitur per inventionem aequationis, componitur per constructionem ejusdem, sed non prius solvitur quam constructionis enunciatio ac demonstratio tota componitur, aequatione neglecta." MP, 7, p. 306.

[8] MP, 7, p. 251. "Nam problemata fere omnia naturalem aliquem habent solvendi modum ... unde factum puto quod Veteres qui ad Compositionem collimabant simpliciores conclusiones assequi solerent, quam recentes qui magis colunt Algebram." MP, 7, p. 250.

[9] [aa]another version. MP. 7, p. 255. "ad solutionem problematis per solam Algebram ... vita hominis non sufficeret [a]nec Herculis patientia nec anni Methusalem sufficerent[a]." MP, 7, p. 254.

drawback is that it does not reveal an elegant and simple composition to the geometer.[10] As Newton stated in commenting on Antonio Hugo de Omerique's work entitled *Analysis Geometrica* (1698),

> [T]herein is laid a foundation for restoring the Analysis of the Ancients wch is more simple, more ingenious & more fit for a geometer than the Analysis of the Moderns. For it leads him more easily & readily to the composition of Problems, & the Composition wch it leads him to is usually more simple & elegant than that wch is forct from Algebra.[11]

Analysis, whether modern algebraic or ancient porismatic, is in any case, in Newton's opinion, unfit to be presented as demonstrative. Analysis, even the ancient analysis, is a complex procedure made of trials and errors and constitutes the process of discovery adopted by the skilled mathematician, who will begin with a conjecture, draw consequences, introduce additional lemmas, adapt the initial hypotheses, until he reaches some already achieved result on which the synthesis or composition can be built. Further, the synthesis, or composition, is not a mere reversal of the steps followed in the analysis; composition becomes true demonstration only when all traces of the heuristic, complex, and tortuous analytical process are eliminated.[12]

It is interesting to note that Newton's wholehearted endorsement of the superiority of the ancient method of resolution and solution over the algebra employed by the moderns; his efforts in restoring ancient porismatic analysis; his program of embedding his geometrical methods (projective, organic, and fluxional) within a historical reconstruction that views his mathematics in line with such ancient tradition; his distancing himself from the heuristic, nonrigorous, and esthetically defective mathematical methods of his contemporaries all occur in a period of his life in which he was deeply involved in the myth of the ancients' *prisca sapientia*. As I have tentatively argued in my *Reading the Principia*, Newton's admiration for ancient geometry might be related to his conviction of being, as a natural philosopher, a privileged interpreter of the Book of Nature and, as a pious Christian, a

---

[10] This point is made particularly clear in Brigaglia, "La Riscoperta dell'Analisi e i Problemi Apolloniani" (1995).

[11] Bodleian. New College MS 361.2, f. 19r discussed in Pelseneer, "Une Opinion Inédite de Newton sur l'Analyse des Anciens" (1930).

[12] In a draft Preface to the "Geometriae Libri Duo" Newton opined, "Whence it comes that a resolution which proceeds by means of appropriate porisms is more suited to composing demonstrations than is common algebra. [a]By present day analysis we arrive promptly at equations, but it is in most cases difficult to derive constructions and demonstrations of the better quality from the equations than happily to accomplish them by means of a complete resolution and composition[a]." MP, 7, p. 261. "Unde fit ut Resolutio quae per debita Porismata procedit sit aptior componendis demonstrationibus quam Algebra vulgi. [a]Hac analysi prompte pervenitur ad aequationes, sed ex aequationibus constructiones et demonstrationes melioris notae derivare utplurimum difficilius est quam per resolutionem et compositionem totam feliciter perficere[a]." MP, 7, p. 260. [aa]another version.

privileged interpreter of the Book of Scripture. Newton began looking at ancient texts not only for mathematical interests. As historians of Newton's chemistry and theology have shown, he was driven by the desire to restore an ancient knowledge of alchemy and religion; it seems probable that this attitude matured in the 1690s. It is plausible that in Newton's mind the restoration of the lost books of the ancient geometers of Alexandria was in resonance with his attempt to reestablish the lost wisdom of the ancients.[13]

But Newton's option for the ancient method of analysis and synthesis, and for a mechanically based geometry—notwithstanding his insistence on the desire to conform to ancient exemplars—was rooted in philosophical agendas motivated by mid-seventeenth-century debates on the nature of mathematics and its relation with natural philosophy.

14.2 Realism and Constructivism

Recent studies devoted to the history of British mathematics have related the mathematical work of Hobbes, Barrow, and Newton to the empiricist philosophy pursued in England and Scotland. Pycior underlined how the preference for geometry over algebra manifested by many British mathematicians was the result of a quest for an empirically based mathematics. In his study on Colin Maclaurin, Sageng defined the British fluxional school as dominated by mathematical empiricism. Sepkoski studied a nominalist and constructivist tradition in mathematics spanning from Pierre Gassendi to George Berkeley, via Barrow, Hobbes, and Newton; the tradition was also highlighted by Garrison.[14] According to Sepkoski, the philosophy of mathematics endorsed by Newton can be termed physicalist insofar as it implies a belief that mathematical representations should be closely aligned with the properties of physical bodies and their motions. The different perspectives taken by Pycior and Sageng on the one side and Sepkoski and Garrison on the other reveal that it is very hard to pigeonhole Newton as a philosopher of mathematics. In the *Preface* to the *Principia*, Newton stated that geometrical objects must be conceived of as generated mechanically (§13.1), and this seems in line with a constructivist position. On the other hand, in the introduction to *De Quadratura*, Newton stated that the

---

[13] Guicciardini, *Reading the Principia* (1999), pp. 30–1. The literature concerning Newton's alchemy and theology is vast. See, for instance, Knoespel, "Interpretive Strategies in Newton's *Theologiae Gentilis Origines Philosophicae* (1999); Snobelen, "'The True Frame of Nature" (2005); Figala "Die Exakte Alchemie von Isaac Newton" (1984); Principe, "The Alchemies of Robert Boyle and Isaac Newton" (2000); Newman and Principe "Alchemy vs. Chemistry" (1998).

[14] Garrison, "Newton and the Relation of Mathematics to Natural Philosophy" (1987); Pycior, *Symbols, Impossible Numbers, and Geometric Entanglements* (1997); Sageng, *Colin MacLaurin and the Foundations of the Method of Fluxions* (1989); and Sepkoski, *Nominalism and Constructivism in Seventeenth-Century Mathematical Philosophy* (2007).

mathematical quantities he considered in the method of fluxions have an existence in *rerum natura*, a pronouncement that suggests a realist view of mathematics (§9.5).

This last theme was to become a leitmotif during the priority controversy with Leibniz; the method of fluxions was presented by Newton's defendants as endowed with ontological content (see part VI). Newton claimed that fluents and fluxions are really exhibited in *rerum natura*, whereas Leibnizian infinitesimals do not exist. But this is not to say, as Sepkoski convincingly argued, that for Newton the geometrical representations themselves were the ontologically real entities they describe; rather their manner of description is closely related to the real world that we perceive. Mathematical geometrical magnitudes are constructed by human faculties, but they are constructed in a way that is not arbitrarily detached from empirical experience.

Newton often insisted also on the fact that the magnitudes of the fluxional method are accessible to perceptual experience. In its mature form, the fluxional method whenever it is possible employs, according to Newton, sensible finite magnitudes.[15] The generation of magnitudes can be performed by "God, nature or any technician"[16] (see chapter 13). Geometry mimics the constructive powers of God. A passage from the work known under the title of "De Gravitatione et Aequipondio Fluidorum," a metaphysical manuscript whose dating is controversial, resonates with Newton's pronouncements on the relation between human and divine geometry:

> [T]he analogy between the divine faculties and our own may be shown to be greater than has formerly been perceived by philosophers. ... Moreover, in moving bodies we create nothing, nor can we create anything, but we only simulate the power of creation. ... if anyone, however, prefers this our power to be called the finite and lowest level of the power which makes God the creator, this no more detracts from the divine power than it detracts from his intellect that intellect belongs to us in a finite degree too.[17]

Mechanically described figures, curves in particular, are thus generated by a faculty that mimics nature and God. In this sense Newton stated that mechanically

[15] In the *De Quadratura*, he writes, "For fluxions are finite quantities and real, and consequently ought to have their own symbols; and each time it can conveniently so be done, it is preferable to express them by finite lines visible to the eye rather than by infinitely small ones." MP, 8, pp. 113, 115.

[16] MP, 7, p. 286.

[17] Add. 4003, ff. 23–4. The title consists of the opening words of this manuscript. Translation by Christian Johnson in Newton, *Philosophical Writings* (2004), p. 30. "ut Analogiam inter nostras ac Divinas facultates majorem esse ostenderem quàm hactenus animadvertêre Philosophi. ... Sed praeterea movendo corpora non creamus aliquid nec possumus creare sed potestatem creandi tantùm adumbramus. ... Siquis autem maluit hanc nostram potestatem dici finitum et infimum gradum potestatis quae Deum Creatorem constituit, hoc non magis derogaret de divina potestate quàm de ejus intellectu derogat quod nobis etiam finito gradu competit intellectus." Newton, "De Gravitatione et Aequipondio Fluidorum" (1962), p. 108.

generated curves, surfaces, and solids are "geneses which exist *in rerum natura*." The fact that one can study them only through approximation techniques, such as infinite series, arises from the limitations of the human intellect, for "we, mere men possessed only of finite intelligence, can neither designate all their terms nor so grasp them as to ascertain exactly the quantities we desire from them."[18]

It is interesting to note that Newton's God is conceived of as a geometer rather than as a calculator. A supreme calculator would be able to create a world that functions according to the necessity of an algorithm. A supreme geometer, instead, freely acts in nature, creating regions of impenetrable space, moving bodies, and intervening in the absolute space and time in which he abides.

Newton's rejection of Descartes' methodological prescriptions concerning curves and his preference for a mechanically based geometry are thus intertwined with many facets of his views on mathematics: his geometrical classicism, his constructive and empiricist conception of mathematical objects, and even his epistemological, ontological, and theological concerns. The next section examines the relations that Newton envisaged between the method of analysis and synthesis, a mechanically based geometry, and the mathematization of force in some of his later writings.[19]

14.3 Analysis and Synthesis in Natural Philosophy

14.3.1 Newton's Pronouncements

Pronouncements concerning connections between the mathematical method of analysis and synthesis and the demonstrative practice in natural philosophy occur explicitly both in the *Principia* and in the *Opticks*. The first explicit reference to the use of the dual method of mathematicians in natural philosophy can be found in a draft preface to the first edition of the *Opticks*, which Newton wrote in 1703-1704:[20]

> As Mathematicians have two Methods of doing things wch they call Composition & Resolution & in all difficulties have recourse to their method of resolution before they compound so in explaining the Phaenomena of nature the like methods are to be used & he that expects success must resolve before he compounds. For the explications of Phaenomena are Problems much harder then those in Mathematicks. The method of Resolution consists in trying experiments & considering all the Phaenomena of nature relating to the subject in hand & drawing conclusions from

[18] MP, 2, p. 243.

[19] Pioneering work in this area has been carried out by Henry Guerlac in "Newton and the Method of Analysis" (1973). I am particularly indebted to Guerlac's thoughtful essay.

[20] James E. McGuire discovered and commented on this manuscript. McGuire, "Newton's 'Principles of Philosophy': An Intended Preface for the 1704 *Opticks*" (1970). The definitive study on Newton's use of analysis and synthesis in natural philosophy is Alan Shapiro, "Newton's 'Experimental Philosophy'" (2004).

> them & examining the truth of those conclusions by new experiments & drawing new conclusions (if it may be) from those experiments & so proceeding alternately from experiments to conclusions & from conclusions to experiments until you come to the general properties of things. [& by experiments & phaenomena have established the truth of those properties]. Then assuming those properties as Principles of Philosophy you may by them explain the causes of such Phaenomena as follow from them: wch is the method of Composition. But if wthout deriving the properties of things from Phaenomena you feign Hypotheses & think by them to explain all nature you may make a plausible systeme of Philosophy for getting your self a name, but your systeme will be little better then a Romance.[21]

This first pronouncement on the use of analysis and synthesis in natural philosophy was incorporated in the Latin edition of the *Opticks* (1706) as Quaestio 23. Its English version as Query 31 (1717) is often cited. In this new version Newton added,

> Analysis consists in making experiments & observations & in arguing by them from compositions to ingredients & from motions to the forces producing them & in general from effects to their causes & from particular causes to more general ones, till the Argument end in the most general.[22]

The interesting fact about this last pronouncement is that Newton explicitly draws an analogy between the methods followed in the *Principia* and those adopted in the *Opticks*, two works that are often described as representative of different methodologies. In the *Principia* analysis is a deduction of forces from motions; in the

[21] Add. 3970, f. 480v. The square brackets are Newton's. Discussed in Shapiro, "Newton's 'Experimental Philosophy'" (2004). "Your systeme will be little better then a Romance": probably a reference to Descartes' advice to his readers in the letter to the abbé Claude Picot (the translator of the first French edition of *Principia Philosophiae*) that Newton might have read in Latin translation in Descartes, *Opera Philosophica* (1656), which he possessed. See Harrison, *The Library of Isaac Newton* (1978), no. 506.

[22] For the reader's convenience, I quote in full: "As in Mathematicks so in Natural Philosophy the investigation of difficult things by the method of Analysis ought ever to precede the method of Composition. This Analysis consists in making experiments & observations & in arguing by them from compositions to ingredients & from motions to the forces producing them & in general from effects to their causes & from particular causes to more general ones, till the Argument ends in the most general: The Synthesis consists in assuming the causes discovered & established, as Principles; & by them explaining the Phaenomena proceeding from them, & proving the explanations. In the two first Books of these Opticks I proceeded by Analysis to discover & prove the original differences of the rays of light in respect of refrangibility reflexibility & colour & their alternate fits of easy reflexion & easy transmission & the properties of bodies both opake & pellucid on which their reflexions & colours depend: & these discoveries being proved may be assumed as Principles in the method of Composition for explaining the phaenomena arising from them: an instance of wch Method I gave in the end of the first Book." Add. 3970, f. 286r. Alan Shapiro in "Newton's 'Experimental Philosophy'" (2004), p. 197, observed that Newton's English version of this paragraph served as the basis for Samuel Clarke's translation into Latin, which differs somewhat from Newton's text. See Newton, *Optice* (1706), pp. 347–8.

Opticks it is a deduction from compositions to ingredients (clearly, the experiments carried on in the analytical process concern the decomposition of white light into its components). In both cases one has a deduction of causes from effects. Once the analytical process arrives at some established cause, the synthesis can begin as a deduction of new effects from known causes.

Further, the procedure of deduction from experiments (in the *Opticks*) and from phenomena or observations (in the *Principia*) has the tentative, heuristic, and complex structure of the analytical heuristic method of the mathematicians. Newton could draw a comparison between the experimental method adopted in natural philosophy and the method of analysis of the mathematicians because he placed experimentation within a deductive mathematical procedure (causes, or principles, are not induced but deduced from the phenomena). The analogy with the analytical method of the mathematicians is also justified by the piecemeal and branching structure of the heuristic alternating process of multiple deductions "from conclusions to experiments, and from experiments to conclusions" that characterizes the mathematical natural philosophy that Newton pursued.[23]

After 1703, Newton often insisted on this theme, both in the *Opticks* and in the second edition of the *Principia.* Already in the *Preface* to the first edition, he wrote,

> And therefore our present work sets forth mathematical principles of natural philosophy. For the basic problem of philosophy seems to be to discover the forces of nature from the phenomena of motions and then to demonstrate the other phenomena from these forces.[24]

This concept was reiterated in the *Preface* to the second edition (1713), signed by Roger Cotes:

> [Those whose natural philosophy is based on experiment] proceed by a twofold method, analytical and synthetic. From certain select phenomena they deduce by analysis the forces of nature and the simpler laws of those forces, from which they then give the constitution of the rest of the phenomena by synthesis. This is that incomparably best way of philosophizing which our most celebrated author thought should be justly embraced in preference of others.[25]

Deducing forces from select phenomena is thus proposed as an instantiation of the method of analysis. Knowledge of the existence of a central inverse-square force acting between massive bodies is not achieved via patient Baconian induction, a collection of numerous, carefully conducted experiments. It is rather deduced mathematically. It is such a mathematical deduction, Cotes made clear, that delivers certainty to the conclusions reached in the *Principia*:

---

[23] This point has been made clear by Ihmig in "Newton's Program of Mathematizing Nature" (2005).

[24] *Principles*, p. 382.

[25] *Principles*, p. 386.

> Now it is reasonable to accept something that can be found by mathematics and proved with the greatest certainty: namely, that all bodies moving in some curved line described in a plane, which by a radius drawn to a point (either at rest or moving in any way) describe areas about that point proportional to the times, are urged by forces that tend towards that same point. ... The following rules must also be accepted and are mathematically demonstrated. If several bodies revolve with uniform motion in concentric circles, and if the squares of the periodic times are as the cubes of the distances from the common center, then the centripetal forces of the revolving bodies will be inversely as the squares of the distances. Again, if the bodies revolve in orbits that are very nearly circles, and if the apsides of the orbits are at rest, then the centripetal forces of the revolving bodies will be inversely as the squares of the distances.[26]

This is the gist of the argument for gravitation at the beginning of Book 3 of the *Principia*. The mathematical propositions of the first Book make it possible to deduce the existence of an inverse-square force from phenomena "agreed among astronomers."[27]

The deduction of forces from phenomena is presented by Newton as the analytical stage of mathematical natural philosophy. Once the forces are established, the process is reversed and the synthetic stage begins. Now one deduces phenomena from the forces. The importance of this concept cannot be overestimated; it appeared at the end of the General Scholium (1713) in one of the most famous Newtonian pronouncements about the rejection of hypotheses in natural philosophy.[28] Newton also expanded on the role of the dual method of analysis and synthesis in natural philosophy in the *Intended Preface* (late 1710s) to the *Principia* (§13.2).[29]

Newton's pronouncements on the use of analysis and synthesis in natural philosophy, which first appeared in print in the Latin *Optice* (1706), were probably an ingredient of his reply to the criticisms of the *Principia* by the continentals (Leibniz among them). To the many who had accused him of having introduced into natural philosophy either an occult cause or a force (gravitation) whose mechanism of action remained unexplained, he wished to reply that gravitation had been deduced from phenomena and that its existence was therefore mathematically certain. As had been the case with his early researches on optics, Newton cushioned what he considered the indisputable certainty of his natural philosophy by profiling its methods along the demonstrative practices of the mathematicians.

---

[26] *Principles*, pp. 387–8.

[27] *Principles*, p. 388. The propositions referred to by Cotes are Proposition 2, Book 1, where it is proved that the area law implies a central force; Proposition 4, Book 1, which proves that for circular orbits Kepler's third law implies an inverse-square force; and Proposition 45, Book 1, which demonstrates that for elliptical orbits the quiescence of the apsidal line implies an inverse-square (or an elastic) force. These are the propositions deployed in the analytical demonstration of gravitational force from phenomena.

[28] *Principles*, p. 943.

[29] MP, 8, pp. 442–59. *Principles*, pp. 49–54.

14.3.2 Understanding Newton's Pronouncements

My purpose in this section is to decode Newton's pronouncements on the canon of problem resolution (analysis) and solution (synthesis) for a mathematized natural philosophy. It is therefore appropriate to summarize some salient aspects of Newton's mathematical methodology that were previously discussed.

Newton wanted to inject certainty into natural philosophy via geometry. From his viewpoint, algebra was not endowed with the certainty that characterizes geometry. He often repeated that geometrical objects, such as plane curves, are better understood if the reason of their genesis is known. Recourse to equations can be helpful in the context of discovery, but in the end equations must be neglected. Therefore, Newton conceived geometrical objects as generated by mechanical devices. Most notably, curves are generated by tracing mechanisms.

In his practice of geometrical analysis Newton typically faced locus problems (e.g., the Pappus problem) in which it is required to construct a curve that satisfies a number of given conditions. The analysis begins from the assumption that the sought curve is given, and proceeds by deduction until constructions given either by postulate or by previously accomplished constructions are obtained. In short, one assumes the sought curve as given and deduces the tracing mechanism that generates it. In the synthetic stage one starts instead from the given conditions and, via allowed constructive tools (postulates or already achieved constructions) obtains the required curve.[30]

When the aim is to geometrize the natural philosophy of force and motion, instead of curves one considers trajectories, instead of tracing mechanisms one considers forces. From Newton's viewpoint, mechanics provided geometry with its subject matter, and it did so with a rich variety of mechanical constructions. Such constructions could be produced by God, nature, or human agency.[31] Therefore, Newton's fluents can be thought of as produced either by human-made tracing mechanisms, or by natural forces. Indeed, conic sections can be conceived of as traced by rotating rulers as well as by physical causal agents, such as inverse-square central forces. Newton remained trapped in the fascination of this ambiguity between human-made construction and reality, an advantageous ambiguity in the sense that it allowed him a seemingly smooth transition from constructive geometry to the mathematical natural philosophy of motions, velocities, and accelerations. Indeed, his peroration in favor of the method of analysis and synthesis in natural philosophy cannot be understood too literally. It is a peroration that emerged in his mature

---

[30] For an example, see §5.4.5.

[31] "Both the genesis the subject-matter of geometry ... and the fabrication of its postulates pertain to mechanics. Any plane figure executed by God, nature or any technician [*a Deo Natura Artifice quovis confectas*] you are to measure by geometry by the hypothesis that they are exactly constructed." MP, 7, p. 289.

years, a period in which it could serve a rhetorical purpose in his polemic against the method of the Cartesians, and Leibniz in particular.

Newton claimed that in natural philosophy in the analytical stage one has to assume the trajectory as given (the phenomenon) and deduce the tracing causal agent (the force). Once this is done, the synthesis reverses the process: now, given the forces, one mathematically deduces the trajectory. The analogy between the analytical/synthetic process applied to mechanically generated geometrical objects, whereby one deduces tracing mechanisms from curves (analysis) and curves from tracing mechanisms (synthesis), and the double implication of forces from phenomena and vice versa of phenomena from forces, would guarantee the applicability of geometrical methods to natural philosophy and therefore the transfer of certainty from geometry to planetary theory and optics. An algebraic approach to natural philosophy would not yield such a success. As algebra was inappropriate in studying the nature of curves, so it was in studying the nature of trajectories. As Colin Maclaurin wrote,

> In order to proceed with perfect security, and to put an end for ever to disputes, [Newton] proposed that, in our inquiries into nature, the methods of *analysis* and *synthesis* should be both employed in a proper order.[32]

Note that in order to guarantee an ontological content to the forces deduced from trajectories, Newton had to introduce his famous concepts of absolute time and space. The phenomena from which forces are deduced are accelerations measured as deviations from inertial motion.[33] It was essential for Newton that such deviations be measured as a function of absolute time and space (in modern terms, in an inertial reference frame). Accelerations measured in an arbitrary reference frame would imply deductions of forces that are not real because they are not located in real bodies that cause them. So, for instance, a reference frame at rest with the fixed stars allows one to measure accelerations of planets from which it is possible to deduce the existence of a force exerted by the sun on the planets. In a reference frame at rest with the earth, one would measure planetary accelerations from which apparent forces acting on the planets would be deduced, forces not endowed with ontological content. A kinematic geometry of fluent magnitudes is the right language in natural philosophy because it enables mimicking of real motions, in real space and real time, and therefore permits the deduction of real forces. The arbitrary and conventional character of algebra would have more difficulty answering the exigency of ontological content, at least this seems to be what Newton might have claimed. Algebraically speaking, all reference frames are equivalent via suitable transformations of coordinates.

[32] Maclaurin, *An Account of Sir Isaac Newton's Philosophical Discoveries* (1748), p. 9. See Guerlac, "Newton and the Method of Analysis" (1973), p. 378.
[33] See, for instance, the segment QR in Proposition 6, Book 1 (§10.2.4).

Note also that once the forces are identified, one assumes the force law and *deduces* new phenomena; this is the compositive, synthetic part of the method. The synthetic part can be ampliative in our knowledge of the world, that is, it can add to that which is already known. As Newton stated in a manuscript related to the famous Query 23/31 of the *Opticks*,

> I derived from it [the inverse-square law] all the motions of the heavenly bodies & the flux & reflux of the sea, shewing by mathematical demonstrations that this force alone was sufficient to produce all those Phaenomena, & deriving from it (a priori) some new motions wch Astronomers had not then observed but since appeare to be true.[34]

Notwithstanding the difficulties that plague this bold methodological project (§14.3.3), considering it allows understanding of several typical aspects of Newton's mathematized natural philosophy, most notably, a better understanding of the way in which he deployed experiments as crucial, or select phenomena in the analytical deduction of forces. As Garrison stated,

> For Newton induction was merely a generalization from an experimental configuration. Repetitions of an experiment were no more necessary for Newton than repetitions of a geometrical configuration would be for the geometrician. This almost anti-inductivist view of induction was as much misunderstood by Hooke, who appealed in his controversy with Newton to "many hundreds of trials," as it has been by contemporary positivistic philosophers of science.[35]

Here Garrison is perhaps too optimistic in his claim of having captured all the facets of Newton's methodology. Newton's experimental practice, both in the *Principia* and in the *Opticks*, is too complex to be labeled anti-inductivist, and Newton himself in his late years showed growing awareness of the fact that experimental philosophy could not be reduced so easily to certainty via an imitation of the mathematicians' methods. This said, I believe that Garrison has identified one important aspect of Newton's views concerning the use of crucial experiments. Newton did not aim at a patient collection of phenomena to be used as a basis for probabilistic inductive generalization, but rather he wished to identify a well-chosen phenomenon that reveals the action of force to the mathematically trained natural philosopher, whose main business is the certain deduction of forces.

14.3.3 Difficulties

How successful was Newton in refuting probabilism and fostering certainty in natural philosophy by deploying the method of analysis and synthesis, and by subsuming

[34] MS Add. 3970. Commented on in Guerlac, "Newton and the Method of Analysis" (1973), p. 386.

[35] Garrison, "Newton and the Relation of Mathematics to Natural Philosophy" (1987), p. 621.

geometry to mechanical practice? I point out here a few difficulties that were perceived by Newton's contemporaries, and I believe also by Newton himself, who, as Alan Shapiro has shown, drifted in his maturity toward a more favorable acceptance of experimental philosophy as an inquiry distinct from mathematics, yielding probable truths and open to failure.[36]

A first difficulty concerns the fact that it is often problematic to map the method of analysis and synthesis, as described in Newton's pronouncements in Query 23/31 and in Cotes's Preface, into Newton's demonstrative practices in natural philosophy.[37] An even cursory inspection of the problems that Newton tackled in the *Principia* reveals that many of them have to do with the analytical (in the strictly mathematical sense) determination of effects from forces, not of forces from effects. One can think of the so-called inverse problems of central forces whose purpose is to determine the trajectory of a body, given known forces acting on it and initial conditions (position and velocity). Such problems are resolved analytically by assuming the sought trajectory as given and deducing consequences from this assumption.[38] Many of the problems that Newton claimed to have resolved by the use of new analysis were problems in which the task was to determine the trajectory given the force, not the force given the trajectory, as in the method of analysis referred to in Query 23/31 and similar passages. This difficulty, however, can be overcome by noting that the pronouncements on the method of analysis and synthesis in natural philosophy do not refer to the mathematical propositions of the first two books of the *Principia* but to the deduction of universal gravitation in the third book. Newton drew a sharp distinction between the mathematical level followed in the first two books and the physical (or philosophical) level of the last book. As he wrote at the beginning of the third book,

> In the preceding books I have presented principles of philosophy that are not, however, philosophical but strictly mathematical—that is, those on which the study of philosophy can be based. These principles are the laws and conditions of motions and of forces, which especially relate to philosophy. ... It still remains for us to exhibit the system of the world from these same principles.[39]

This concept was already made clear in the *Preface* to the first edition:

> [F]or the basic problem of philosophy seems to be to discover the forces of nature from the phenomena of motions and then to demonstrate the other phenomena from

[36] Alan Shapiro, "Newton's 'Experimental Philosophy'" (2004).

[37] For the difficulties in discerning use of the method of analysis and synthesis in the *Opticks*, see Shapiro, "Newton's 'Experimental Philosophy'" (2004).

[38] Newton took both a geometrical and an algebraic approach to the inverse problem of central forces in the *Principia* (§10.2.4, §10.2.7). It is particularly interesting to read Corollary 1, Propositions 11–13, Book 1, as related to the analytical procedures deployed by Newton in tackling locus problems.

[39] *Principles*, p. 793.

> these forces. It is to these ends that the general propositions in books 1 and 2 are directed, while in book 3 our explanation of the system of the world illustrates these propositions. For in book 3, by means of propositions demonstrated mathematically in books 1 and 2, we derive from celestial phenomena the gravitational forces by which bodies tend toward the sun and toward the individual planets. Then the motions of the planets, the comets, the moon, and the sea are deduced from these forces by propositions that are also mathematical.[40]

Therefore, in the first two books the mathematical method of analysis and synthesis is applied to problems that relate forces to phenomena (mainly trajectories), which are seen as purely abstract mathematical objects and are not necessarily instantiated in nature. In the first two books the relation between forces and phenomena is purely inferential and lacks a causal dimension. Here in general Newton sought for double implications of the form the force is directly proportional to distance if and only if the trajectory is an ellipse and the area law is valid for the ellipse's center; or, the force is inverse-square if and only if the trajectory is a conic section and the area law is valid for a focus. Then, in the third book, the mathematical results obtained in the first two books are applied in order to deduce the real forces acting in nature from observed phenomena, most notably from planetary trajectories measured in an inertial reference frame. Newton claimed that the deduction from observed trajectories to real forces was analogous in physics to the analytical process followed in mathematics, whereas the deduction of observed trajectories from real forces was analogous to mathematical synthesis.

A second difficulty concerns an epistemological difference between geometrical objects generated by tracing mechanisms operated by a technician and trajectories generated by natural forces. Whereas the tracing mechanisms are under the control of the geometer, the forces acting in nature are in principle inaccessible to direct knowledge; they can only be inferred by deduction from the phenomena. As Ducheyne has shown, this asymmetry between the order of nature and the order of knowledge was amply discussed in Aristotelian textbooks accessible to Newton. Indeed, Newton's discourse on analysis and synthesis was influenced not only by the mathematical Pappian tradition but also by a philosophical tradition dating back to Aristotle's *Posterior Analytics*. In the jargon of Aristotelian natural philosophy one would say that in the order of things (*ordo naturae*) causes come first and effects follow from them, whereas in the order of knowing one notices effects first and from them tries to infer the causes. Such inference from known effects to causes was analysis as *resolutio* or *demonstratio quia*. After the analysis (sometimes defined as a process of induction) the Aristotelian philosopher was expected to infer from causes

[40] *Principles*, p. 382. See also the final scholium to Section 11, Book 1: "Mathematics requires an investigation of those quantities of forces and their proportions that follow from any conditions that may be supposed. Then, coming down to physics, these proportions must be compared with the phenomena." Ibid, p. 588.

to effects, producing a synthesis, a *compositio* or *demonstratio propter quid*.[41] The analogy between the methods of the geometers and those of the natural philosophers breaks down at this level. The analytical method of the mathematicians is a process in which one obtains what is known or given from what is unknown or searched for, whereas in Aristotelian natural philosophy analysis is a process in which one abtains what is unknown (but first in the order of nature) from what is known. Newton, when writing about analysis and synthesis in his late writings, referred to two different traditions. He referred to the mathematical tradition, eminently represented by Pappus, in writings like "Geometriae Libri Duo," and to the Aristotelian tradition in the Query 23/31 of the *Opticks*. It seems to me that Newton conflated these two different conceptions of analysis and synthesis (the Pappian and the Aristotelian) as a rhetorical move aimed at defending the certainty of his natural philosophy.

The asymmetry between geometry and natural philosophy was discussed by Hobbes and Barrow. Hobbes had maintained a materialistic view of geometry in order to claim its scientific status. Geometry is scientific because its objects are generated by mechanisms that we know and control:

> "But," you will ask, "what need is there for demonstrations of purely geometric theorems to appeal to motion?" I respond: "First, all demonstrations are flawed, unless they are scientific, and unless they proceed from causes, they are not scientific. Second, demonstrations are flawed unless their conclusions are demonstrated by construction, that is, by description of figures, that is, by the drawing of lines. For every drawing of a line is motion: and so every demonstration is flawed, whose first principles are not contained in the definitions of motions by which figures are described."[42]

> The science of every subject is derived from a precognition of the causes, generation, and construction of the same, and consequently where the causes are known, there is place for demonstration, but not where the causes are to seek for. Geometry therefore is demonstrable, for the lines and figures from which we reason are drawn and described by ourselves. ... But because of natural bodies we know not the construction, but seek it from the effects, there lies no demonstration of what the causes be we seek for, but only of what they might be.[43]

These themes were dealt with long before in the mid-seventeenth-century debates *de certitudine mathematicarum* initiated by Piccolomini.[44] Further, Hobbes declared

---

[41] See Ducheyne, "Newton's Training in the Aristotelian Textbook Tradition" (2005).

[42] Hobbes, *De Principiis* (1666), in *Opera Philosophica* (1839–1845), 4, p. 421. Translated and commented on in Jesseph, *Squaring the Circle* (1999), p. 135.

[43] Hobbes, *Six Lessons to the Professors of the Mathematiques* (1656), in *English Works* (1839–1845), 7, p. 184.

[44] On Hobbes's involvement with the *de certitudine mathematicarum* debate and his idea of mathematics as a science of causal relations, see Jesseph, *Squaring the Circle* (1999).

in the opening chapter of *De Corpore*, which Newton knew since his early days as a student in Cambridge,[45] that philosophy is "knowledge of effects or appearances, as we acquire by true ratiocination from the knowledge we have first of their causes or generation: And again, of such causes or generations as may be from knowing first their effects."[46] This statement has some resemblance with several Newtonian definitions of the aims of natural philosophy.[47] Similar observations can be found in Barrow, who—echoing arguments advanced by Piccolimini's critics—advanced the idea of geometry as a science of causes.[48] Both Hobbes and Barrow showed awareness of the asymmetry between geometry and natural philosophy: unlike geometrical constructive tools, natural causes are secondary in the order of knowing.

Barrow showed awareness of a third difficulty that undermines Newton's position: geometrical causal relations lack the uniqueness required by physical causes. As Newton was to do after him, Barrow equated the tracing mechanisms employed in the organic generation of curves to causal mechanisms.[49] But Barrow also emphasized that the same curve can be conceived of as generated by different causal mechanisms. As Malet made clear, this led Barrow to understand geometry as a science of possible worlds and to relate this view to a voluntaristic conception of God, who chooses one of the possibilities envisaged by the geometer as the real world.[50] Barrow's epistemology of mathematics is an example of the relation between constructivism and voluntarist theology, analyzed by Funkenstein.[51]

---

[45] See McGuire and Tamny's commentary in Newton, *Certain Philosophical Questions* (1983), p. 219.

[46] Hobbes, *English Works* (1839–1845), 1, p. 3.

[47] See, for example, the *Preface* to the *Principia*, namely, "[F]or the basic problem of philosophy seems to be to discover the forces of nature from the phenomena of motions and then to demonstrate the other phenomena from these forces." *Principles*, p. 382.

[48] "But those who study to detract not from the Certitude and Evidence, but from the Dignity and Excellence of the Mathematics do bring another Device. For they attempt to prove that mathematical Ratiocinations are not Scientific, Causal, and Perfect, because the Science of a Thing signifies to know it by its Cause.... it is plain that Mathematical Demonstrations are eminently Causal, from whence, because they only fetch their Conclusions from Axioms which exhibit the principal and most universal Affections of all Quantities, and from Definitions which declare the constitutive Generations and essential Passions of particular Magnitudes." Barrow, *The Usefulness of Mathematical Learning* (1734), pp. 80, 83.

[49] "[T]hey [the Mathematicians] assign Generations or Causes easy to be understood and readily admitted to all; they preserve a most accurate Order, every Proposition immediately following from what is supposed and proved before, and reject all Things howsoever specious and probable which cannot be inferred and deduced after the same manner." Barrow, *The Usefulness of Mathematical Learning* (1734), pp. 65–6.

[50] Malet, "Isaac Barrow on the Mathematization of Nature" (1997).

[51] Funkenstein, *Theology and the Scientific Imagination* (1986). For a recent appraisal, see Sepkoski, *Nominalism and Constructivism in Seventeenth-Century Mathematical Philosophy* (2007).

The epistemological asymmetry between known causal tracing mechanisms versus in principle unobservable forces, and between the uniqueness of physical causes versus in principle multiple tracing mechanisms—highlighted by Barrow and Hobbes, who held views on geometry and mechanics so similar to Newton—casts a shadow on Newton's use of the dual method of analysis and synthesis as a way out of the Cartesian or Baconian probabilisms (see chapter 2).

As is well known, the Cartesian and Leibnizian physici accused Newton of having built a mathematical structure devoid of physical content. From their point of view, Newton's dual method was exactly what he claimed for it: a mathematical deduction and, as such, removed from a true physical explanation.[52] Huygens's skepticism was widely shared on the Continent:

> I have great esteem for his [Newton's] knowledge and subtlety, but, in my opinion, he has made a poor use of them in most of this work [the *Principia*], when the author researches things which have little utility, or when he builds on such an unlikely principle as that of attraction.[53]

The phenomena Newton referred to as the starting point of his analytical deductive process are mathematical abstractions (e.g., planets mathematically modeled as point masses tracing ellipses according to Kepler's area law) not observed in the real world. According to Newton's physics of gravitation, real planetary motions are a much more complicated phenomenon, one that did not find a simple place in the dual scheme of analysis and synthesis. Newton did not endorse a Keplerian or Pythagorean faith in a simple mathematical structure underpinning natural phenomena. In the *Principia* neither are forces deduced from real phenomenal planetary motions, nor are these motions exactly deduced from simple inverse-square forces.[54] As Cohen and Smith have amply demonstrated, Newton made brilliant use of the deviations of observed phenomena from the predictions of his mathematical models. It is exactly his success in systematically controlling and reducing these deviations by the construction of successive models which approximate the recalcitrant phenomena better and better that corroborated his theory of gravitation.[55]

---

[52] On the ontological commitments implied by Newton's concept of force, see Janiak, "Newton and the Reality of Force" (2007). Janiak also published *Newton as Philosopher* (2008), which unfortunately I did not see printed during the composition of the present work.

[53] "J'estime beaucoup son scavoir et sa subtilité, mais il y en a bien de mal emploié à mon avis, dans une grande partie de cet ouvrage lors que l'autheur recherche des choses peu utiles, ou qu'il batit sur le principe peu vraisemblable de l'attraction." Huygens, *Oeuvres* (1888–1950), 10, p. 354.

[54] Newton was quick to realize this fact, as he noted in 1684 in a revised version of "De Motu Corporum in Gyrum": "[ut] planetae nec moveantur in Ellipsibus exacte neque bis revolvant in eadem orbita. Tot sunt orbitae Planeatae cujusque quot revolutiones, ut fit in motu Lunae. ... Tot autem motuum causas simul considerare et legibus exactis calculum commodum admittentibus motus ipsos definire superat ni fallor vim omnem humani ingenij." MP, 6, p. 78.

[55] Cohen, *The Newtonian Revolution* (1980); Smith, "The Methodology of the *Principia*" (2002).

These themes have been debated since the publication of the *Principia* and are still the object of intense study by Newtonian scholars. They belong to a level of discourse that I cannot reach from the standpoint of this book and because of the limitations of my stature as a scholar. I hope, nonetheless, that this overview of Newton's mathematics has provided enough information about his method of analysis and synthesis to allow appreciation of the language he used and the aims he had in mind when he addressed himself so passionately to his contemporaries as a defender of the dual method of mathematicians in natural philosophy.

VI Against Leibniz

When Newton confronted Leibniz in the dispute over priority, he was concerned with building up forensic and historical documentation, most notably *Commercium Epistolicum* (1713) and its anonymous "Account" (1715), whose purpose was to prove Leibniz's plagiarism. Many historians have amassed evidence about Newton's obsessive approach to the priority dispute, his lack of fairness, and his egotism. It is also clear that Leibniz had to be opposed for a series of very solid reasons having to do with philosophical and even political issues. The German, who was employed by the Hanover family, after the accession of George I to the throne (August 1, 1714) wished to move to London as Royal Historian. The idea of having in England such a towering intellectual who defended a philosophical view that contradicted Newton's voluntarist theology and who promoted the reunification of the Christian churches was anathema for Newton and his supporters. Part VI disregards all such intertwined issues and focuses on the priority dispute in order to put Newton's views concerning mathematical method and certainty in perspective.

Newton's aim was not only to prove Leibniz's plagiarism. He also wished to highlight the superiority of his method over Leibniz's calculus. The mathematical program that Leibniz promoted with so much success was at odds with Newton's convictions concerning mathematics. Newton therefore defended positions that have deep roots in his protracted opposition against Descartes' canon of analysis and synthesis and the modern mathematicians. He could scarcely fathom the depth of the views concerning mathematics defended by his German opponent. For him, Leibniz's calculus was only "for finding it out," a heuristic analytical symbolism devoid of scientific character. Newton claimed that he himself possessed a synthetic version of the method of fluxions that was well grounded in geometry and in the nature of things.

Further, when comparing the two analytical tools, Newton focused on the rules for squaring curves via approximations that constituted the climax of his *inverse* method of fluxions. Leibniz based his claims as a discoverer of the calculus on his rules for the *direct* differential calculus. For Leibniz, too, the calculus was a heuristic tool, but he valued the systematic use of symbolism and its deductive structure much more than Newton did. The rules of the differential calculus, because of their simplicity and logical priority, were conceived of by Leibniz as playing a major role in a broader logico-philosophical program aimed at the construction of a universal language. By contrast, Newton never attributed great value to the discovery of the rules of the direct method. These diverging orientations may explain the different strategies adopted by Newton and Leibniz during the priority dispute. Newton's policy, which was embodied in *Commercium Epistolicum* (1713), has often been considered lacunose or even contradictory, but such evaluations are the result of a misunderstanding of his agenda.

In order to place the controversy with Leibniz in context, it is important to consider not only the pronouncements that Newton made after its inception but also the strategies he followed after the 1670s in order to spread knowledge about his mathematical discoveries. Chapter 15 surveys the main events related to the controversy, chapter 16 discusses the circulation of Newton's mathematical manuscripts and his mathematical correspondence, and chapter 17 reviews the editorial and authorial policy he followed when printing his mathematical work with the purpose of winning the battle against Leibniz.

15 The Quarrel with Leibniz: A Brief Overview

> Did Newton and Leibniz discover the same thing? Obviously, in a straightforward mathematical sense they did: [Leibniz's] calculus and [Newton's] fluxions are not identical, but they are certainly equivalent. ... Yet one wonders whether some more subtle element may not remain, concealed, for example, in that word "equivalent." I hazard the guess that unless we obliterate the distinction between "identity" and "equivalence," then if two sets of propositions are logically equivalent, but not identical, there must be some distinction between them of a more than trivial symbolic character.
>
> —A. Rupert Hall, 1980

The controversy between Newton and Leibniz has been studied in detail, most notably by Rupert Hall, to whose book *Philosophers at War* I am deeply indebted. In this chapter I give an overview of the main stages of the controversy.

Newton formulated his method of series and fluxions between 1664 and 1666. He continued working and refining the method, obtaining new results and new versions until the 1690s. He also let it circulate in manuscript form, since he was proud of the results he had achieved, but at he first rejected the idea of printing it. The reasons that lay behind Newton's reluctance to print the new method are complex (see chapter 16), but the main cause was likely his awareness that printing a technique that implies the use of infinite series and infinitesimal magnitudes would have involved him in a polemic similar to the one experienced by Wallis after the publication of *Arithmetica Infinitorum* (§7.2). In the 1670s, Newton was fighting a frustrating battle against the naturalists of the Royal Society in order to defend the cogency of his new theory of light. Aspects of this polemic were concerned with the role of mathematics in natural philosophy; for Newton, geometrical philosophers could overcome the uncertainty of the Baconian empiricism in vogue at the Royal Society (see chapter 2). He therefore had to be very careful to avoid a public endorsement of mathematical methods whose cogency might be suspect.

When, in 1672, Leibniz arrived in Paris on a diplomatic mission, he had very little knowledge of advanced mathematics, even though he had concerned himself with combinatorics and universal language, showing a marked talent for abstraction and manipulation of symbols. His encounter with Huygens, the great star of the French Academy, revealed to him how little he knew about recent mathematical

---

Epigraph from Hall, *Philosophers at War* (1980), pp. 257–8.

research. When the German diplomat visited London (in January–March 1673) the impression he had made on Huygens was confirmed: the English mathematicians he met judged him clever but amateurish. He was also suspected of having attributed some results on series, which were well known to the cognoscenti, to himself. His was not a great diplomatic success. Back in Paris, Leibniz carved his way toward the differential and integral calculi. The manuscripts that have survived show an independent path, traced by a towering mind. The result that Leibniz achieved (his notation is still in use) was different from, but at the same time equivalent to, Newton's.

The confrontation between the two competing mathematical methods has been dealt with by many historians of mathematics.[1] It has often been noted that Leibniz showed a greater interest in notation than Newton did; Leibniz devoted a great deal of attention to the basic symbolical rules of the calculus. But considering Leibniz as just a symbol manipulator would be a great simplification. First, his research on mathematical symbols was motivated by broad-ranging philosophical agendas. Second, as early as 1675 he began pondering the foundations of the calculus, devoting profound pages to the problem of the continuum.[2] The result of this research was a foundation of quadrature methods in terms of limits that bears some resemblance to Newton's method of first and ultimate ratios.[3] Leibniz, however, promoted an approach to mathematical research that acknowledged the autonomy of the calculus algorithm from metaphysical questions concerning the continuum and the status of infinitesimals. Therefore, he encouraged his disciples to pursue the development of the algorithm without worrying too much about its meaning. Philosophy was his province of inquiry, but it should not interfere with the free development of mathematical practice.

After moving to Hanover in 1676, Leibniz began printing his mathematical results: in 1682 his series quadrature for π, and two years later a short paper on the differential calculus, which became a landmark in the history of mathematics. In *Nova Methodus* (1684) one finds a statement of the basic rules for the differentiation of the product, power, quotient, and root. Today one immediately recognizes Leibniz's notation and rules as familiar, whereas Newton's method sounds somewhat arcane. In 1686, Leibniz printed another short seminal paper on the integral calculus.

In the meantime Leibniz had been corresponding with Oldenburg on a variety of topics. The secretary of the Royal Society, via Collins, kept him informed about the advances achieved in England and Scotland on infinite series by Newton and

[1] See Hall, *Philosophers at War* (1980); Bertoloni Meli, *Equivalence and Priority* (1993).
[2] Leibniz, *The Labyrinth of the Continuum* (2002).
[3] Leibniz, *De Quadratura Arithmetica* (1993); Knobloch, "Leibniz et Son Manuscrit Inédité" (1989).

James Gregory. Leibniz understood that Newton was a great mind in infinite series expansion. He had no idea, however, that the Lucasian Professor had a method for drawing tangents and squaring curves equivalent to Leibniz's differential and integral calculus. Two letters, known as the *epistola prior* and the *epistola posterior*, which Newton addressed to him via Oldenburg in June and October 1676, opened new vistas.

Leibniz had requested information about Newton's mathematical discoveries, and the latter produced two carefully drafted letters giving information about the method of series and hinting at the method of fluxions. The *epistola posterior* contained two puzzling anagrams (see chapter 16). In 1676 Leibniz had little to learn from these letters and from the decoding of the anagrams, since he had already developed his calculus, but he realized from Newton's results displayed in the *epistolae* that the Englishman had reached an algorithm equivalent in power to his. Indeed, on his way to Hanover from Paris in October 1676, Leibniz visited London for a second time and was able to consult and annotate a copy of *De Analysi* kept at the Royal Society. In June 1677 he replied to Newton, giving all the details of the calculus. Realizing that the presumptuous but sloppy German dilettante, as judged from his visit to London in 1673, could after just four years reach so far must have had a chilling effect on his British correspondents. The correspondence between Newton and Leibniz stopped for a while.

When Newton published the *Principia* he took notice of Leibniz's discovery in a Scholium to Lemma 2, Book 2 (§9.4). The Lemma can be viewed as a first reaction to Leibniz's challenging position in the arena of European mathematics. Indeed, Newton here displayed the rules of the direct method in a way that is not to be found either in *De Analysi* or in *De Methodis*, where he had instead illustrated the rules of the direct algorithm by application to particular examples, namely, the reader had to infer the rules from the particular example, but the rules were not stated in their full generality.[4] Was then Newton giving his method a more systematic form in Lemma 2 in order to emulate Leibniz? This might well be the truth. It should be added that Lemma 2 is an elaboration of "Geometria Curvilinea" (§9.2), which Newton wrote around 1680 (if Whiteside's dating is correct), therefore after receiving Leibniz's 1677 letter.

In the Scholium to Lemma 2, Newton publicly recognized that Leibniz had independently achieved a result similar to his. He mentioned his *epistola posterior* (1676) and Leibniz's reply to him (1677). This Scholium was altered in the third edition of the *Principia* (1726) in a way that was unfavorable to Leibniz; at that time the priority dispute had already erupted and caused much damage. In the original Scholium, Newton, while deciphering the first anagram of the *epistola posterior*,

[4] For instance, from the algorithm illustrated with the example considered in §8.3.2 one can infer the rule of differentiation for the product.

wrote,

> In correspondence which I carried on ten years ago with the very able geometer G. W. Leibniz, I indicated that I was in possession of a method of determining maxima and minima, drawing tangents, and performing similar operations, and that the method worked for surd as well as rational terms. I concealed this method under an anagram comprising this sentence "Given an equation involving any number of fluent quantities, to find the fluxions, and vice versa." The distinguished gentleman wrote back that he too had come upon a method of this kind, and he communicated his method, which hardly differed from mine except in the form of words and notations [a]and the concept of generation of quantities[a]. The foundation of both methods is contained in this lemma.[5]

Notwithstanding Newton's declaration that in Britain an equivalent method had been developed, it is on the Continent that the calculus first began to flourish and expand. During the last decade of the seventeenth century the European journals, most notably *Acta Eruditorum*, began to publish the works by Leibniz and the brothers Jacob and Johann Bernoulli. Pierre Varignon was soon to join this little group with a series of papers printed in the *Mémoires* of the *Académie des Sciences*. These works extended and developed the differential and integral calculus. The continental school was aware of its superiority over the British. As a matter of fact, British mathematicians like John Craig and David Gregory had to consult *Acta Eruditorum* as their source of information on the new calculus, as is evident from Craig's *Methodus Figurarum* (1685) and Gregory's widely circulated manuscript entitled *Methodus Fluxionum*.[6]

Newton continued to reject proposals for printed publication, which in the 1690s were coming from Wallis, who complained about the fact that the "notions of flux-

[5] *Principles*, p. 649n. [aa] added in second edition (1713). In the third edition the Scholium reads as follows: "In a certain letter written to our fellow Englishman Mr. J. Collins on 10 December 1672, when I had described a method of tangents that I suspected to be the same as Sluse's method, which at that time had not yet been made public, I added: 'This is one particular, or rather a corollary of a general method, which extends, without any troublesome calculation, not only to the drawing of tangents to all curve lines, whether geometric or mechanical or having respect in any way to straight lines or other curves, but also to resolving other more abstruse kinds of problems concerning curvatures, areas, lengths, centers of gravity of curves, . . . , and is not restricted (as Hudde's method of maxima and minima is) only to those equations which are free from surd quantities. I have interwoven this method with that other by which I find the roots of equations by reducing them to infinite series.' So much for the letter. And these last words refer to the treatise that I had written on this topic in 1671. The foundation of this general method is contained in the preceding lemma." Newton, *Principles*, pp. 649–50. For comments on this Scholium, see chapter 16.

[6] "Isaaci Newtoni Methodus Fluxionum; ubi Calculus Differentialis Leibnitij, et Methodus Tangentium Barrovij explicantur, et exemplis plurimis omnis generis illustrantur. Auctore Davide Gregorio M. D. Astronomiae Professore Saviliano Oxoniae." Christ Church (Oxford). Other copies are in St Andrews University Library (MS QA 33G8/D12) and in the Cambridge University Library, Macclesfield Collection, Add. 9597.9.3 and Add. 9597.9,4.

ions" were circulating on the Continent "by the name of Leibniz's calculus differentialis."[7] The Savilian Professor was able to obtain just a few fragments from Newton and eagerly included them in his chauvinist *Algebra* (1685) and *Opera* (1693–1699) (§16.4). On the other side of the Channel, one of Leibniz's staunchest defenders, Johann Bernoulli, expressed symmetric complaints, for instance, in his letter to Leibniz concerning Wallis's *Opera* (1693) he stated that too little credit had been given by Wallis to the Leibnizian school.[8]

In January 1697 (N.S.), Bernoulli circulated the brachistochrone problem as a challenge "to the sharpest mathematicians in the whole world." Newton's solution appeared anonymously in the February issue of *Philosophical Transactions*.[9] Newton had probably achieved this solution through a fluxional equation similar to the (unpublished)[10] one he employed for the solid of least resistance. Newton's paper contained a geometrical construction of the curve required (a cycloid) but not the fluxional analysis.[11] In 1699, Fatio, in a work in which he dealt with the brachistochrone, *Lineae Brevissimi Descensus Investigatio Geometrica*, accused Leibniz of having plagiarized Newton's method of fluxions.[12] This episode was dealt with diplomatically, and the case was soon silenced. As Hall wrote, Leibniz received from Wallis the reassurance given to Wallis by the President of the Royal Society, Hans Sloane, that Fatio had obtained the imprimatur of the Royal Society by means of trickery.[13]

In the first decade of the eighteenth century the situation deteriorated. Though Leibniz published his review of *De Quadratura* (1704) anonymously in 1705, it only later provoked Newton's anger when he read it, at Keill's prompting, in 1711. This review could be interpreted as a statement of the inferiority, or even chronological posterity, of Newton's method relative to Leibniz's calculus.[14] Things went

---

[7] *Correspondence*, 4, p. 100.

[8] Leibniz, *Leibnizens Mathematische Schriften* (1849–63), 3, pp. 301, 312, 316–7.

[9] Bernoulli's "Problema Novum, ad Cujus Solutionem Mathematici Invitantur" had already appeared six months before in *Acta Eruditorum* for June 1696, p. 269. Newton's anonymous paper is in *Philosophical Transactions* 19 (1697): 384–9.

[10] MP, 6, pp. 456–80. First published in the Appendix to Motte's English translation of the *Principia* (1729).

[11] Bernoulli began addressing himself "Acutissimis qui toto Orbe florent Mathematicis S[alutem] P[lurimam] D[icit] Johannes Bernoulli." Bernoulli's re-proposal of the challenge (*Acta Eruditorum* for December 1696, p. 560) circulated as a broadsheet. Newton's copy is held at the Royal Society of London. A transcription can be found in MP, 8, pp. 80–5. Newton's fluxional analysis of the brachistochrone problem can be found in the University Library of Cambridge (Add. 3968.41, f. 2r) and was edited by Whiteside in MP, 8, pp. 86–91.

[12] On Fatio de Duillier see Mandelbrote, "The Heterodox Career of Nicolas Fatio de Duillier" (2005).

[13] Hall, *Philosophers at War* (1980), p. 121. Leibniz, *Leibnizens Mathematische Schriften* (1849–63), 3(2), pp. 596–621.

[14] This anonymous review appeared in the *Acta Eruditorum*, (January 1705), pp. 30–6.

to the worse as Leibniz attacked Newton's gravitation theory, eminently in *Essais de Théodicée* of 1710. In the same year, John Keill—it seems without Newton's knowledge—stated in the *Philosophical Transactions* of the Royal Society that Leibniz had plagiarized Newton.[15]

Leibniz was a member of the Royal Society, and so he felt fully entitled to ask for a formal apology for such "most impertinent accusation."[16] His letter was addressed to Sloane on February 21/March 4, 1710/11. Keill was requested by the Royal Society to vindicate himself, which he did, after consulting Newton; he read his reply on May 24, 1711.[17] This reply made things worse for the relationship between Leibniz and the Royal Society. Indeed, Keill, instead of apologizing, produced supposed evidence of the fact that "some specimens" of the Newtonian method had been passed to Leibniz by Newton, Collins, and Oldenburg, giving him "entrance into the differential calculus." Such evidence was in part derived from the papers of Collins, which William Jones had recently acquired. In 1711, Jones published a small booklet of Newton's mathematical tracts and extracts from the correspondence that reinforced Keill's reply, which, being published in the *Philosophical Transactions*, acquired an offending formal character. On December 18/29, 1711, Leibniz required that the Royal Society protect him from the "empty and unjust braying" of such an "upstart" as Keill.[18] Consequently, a committee of the Royal Society, appointed on March 6, 1712, and secretly guided by its president, Isaac Newton, produced a detailed report. *Commercium Epistolicum* was completed just 50 days after the committee's nomination but distributed free of cost only in February 1713 (N.S.). The committee maintained that Newton was the "first inventor" and that "[Leibniz's] Differential Method is one and the same with the Method of Fluxions, excepting the Name and Mode of Notation."[19] It was also strongly suggested that Leibniz, after his visits to London in 1673 and 1676, and after receiving letters and other material from Newton's friends, and in 1676 from Newton himself, had gained sufficient information about the method of fluxions to allow him to publish the calculus as his own discovery, after changing the symbols. It is only after the work of historians such as Fleckenstein, Hofmann, Hall, and Whiteside that we have proof that this accusation was unjust.[20] Newton and Leibniz arrived at equivalent results independently and following different paths of discovery.

Commercium Epistolicum (1713) can be considered Newton's last mathematical work. It is a diplomatic document, formally issued by the Royal Society, based

[15] Keill's "Epistola" was presented in 1708 but printed in 1710.
[16] *Correspondence*, 5, p. 97.
[17] *Correspondence*, 5, pp. 133–41.
[18] "vanae et injustae vociferationes," "cum homine docto, sed novo." *Correspondence*, 5, p. 207.
[19] *Commercium Epistolicum* (1713), p. 121.
[20] Fleckenstein, *Der Prioritätstreit zwischen Leibniz und Newton* (1956); Hofmann, *Leibniz in Paris* (1974); Hall, *Philosophers at War* (1980); Whiteside's commentary in MP, 8.

on Newton's recollections and private archive; letters and manuscripts held in the archives of the Royal Society; excerpts from printed works; and precious manuscripts provided by William Jones, who had acquired the Collins papers, a trove of information on the mathematical activities of the British community in the 1670s.[21] Formally it is the work of an independent committee. Materially, as the manuscripts edited by Whiteside in Volume 8 of *Mathematical Papers* show, it was a work carefully drafted and engineered by Newton, who honed and brought to perfection every detail of it.[22] There is not a word, not a detail, that passed into print without his supervision.

The controversy between Newton and Leibniz which ensued, and which continued after Leibniz's death, involving a number of continental and British mathematicians, theologians, and pamphleteers, is not detailed here. The polemic spanned mathematical, philosophical, religious, and political issues, and Newton's arguments against Leibniz were weapons used on the battlefield of a broad-ranging war. But I confine my attention here to the aspects of the dispute that reveal Newton's methodological convictions concerning mathematics. I am interested in understanding why he chose certain mathematical weapons rather than others, why he used them in such an idiosyncratic way, and why he considered his strategy of attack a convincing one for the expert mathematician.

---

[21] The archival material in possession of William Jones forms part of the Macclesfield Collection recently acquired by the University Library of Cambridge.

[22] MP, 8, pp. 539–60.

16 Scribal Publication, 1672–1699

> Our specious algebra is fit enough to find out, but entirely unfit to consign to writing and commit to posterity.
>
> —David Gregory, 1694

16.1 Newton's Reluctance to Publish

16.1.1 Proposals for Publication

It is well known that Newton, prior to the printing of his mathematical works in the eighteenth century, disseminated knowledge about his mathematical discoveries through correspondence with other mathematicians or via intermediaries such as John Collins and Henry Oldenburg. Correspondence was one of the main vehicles of publication for seventeenth-century mathematicians. However, in his correspondence (§16.3) Newton disclosed only a fraction of his mathematical output; many important results, especially details about proof methods, remained buried in his manuscripts. This aspect of Newton's policy of publication has been extensively researched, especially in studies focused on the priority dispute with Leibniz. Many scholars have elucidated the details and the background of Newton's mathematical letters, most notably the two 1676 *epistolae* to Leibniz, and of *Commercium Epistolicum* (1713), which was supposed to constitute evidence of Leibniz's plagiarism, evidence mainly based upon letters exchanged via Collins in the early days of Newtonian creativity. Little research has been devoted to another aspect of Newton's policy of mathematical publication, that is, the controlled circulation of mathematical manuscripts that Newton engineered in the 1670s, 1680s, and 1690s (§16.2).[1]

Historians of Newton's mathematics cannot avoid feeling disconcerted when they realize that most of the mathematical discoveries achieved by Newton in the late 1660s and early 1670s were printed decades later, basically after the inception of the priority dispute with Leibniz in 1699. These discoveries, especially those concerning

[0] Epigraph from David Gregory's memorandum of a May 1694 visit to Newton. Edinburgh University Library, MS Gregory C42. Translation by Whiteside in MP, 7, p. 196. See also *Correspondence*, 3, p. 385. "Algebram nostram speciosam esse ad inveniendum aptam satis at literis posterisque consignandum prorsus ineptam."

[1] Most of the information on this topic comes from Whiteside's commentary to the eight volumes of Newton's mathematical manuscripts.

the method of series and fluxions, were so innovative that late-seventeenth-century European mathematics would have been different if Newton had been more prompt in sending some of his early manuscripts on the method for publication. To take an example, the priority dispute with Leibniz would have been so avoided.

Newton's early mathematical papers do not consist only of private cluttered notes. He wrote systematically and often in expository mood. In general, throughout his life Newton produced a number of tidy and well-written treatises, which often were not sent to the press. In the nineteenth century Henry Richards Luard, who was inspecting the Portsmouth Collection on behalf of Cambridge University, averred that many of the well-ordered folios had been composed "apparently from the mere love of writing. His [Newton's] power of writing a beautiful hand was evidently a snare to him."[2] Rather than attributing to Newton such schizophrenic behavior, it is more reasonable to assume that his manuscripts were not necessarily meant for the press; they were composed in such beautiful hand in order to be shared, even if in a controlled way, with a closed circle of acolytes. According to my interpretation, it is wrong to conclude too hastily that Newton contemplated the project of printing a mathematical work because its surviving manuscript is written in an impeccable style. After the mid-1670s many of these manuscripts were meant for circulation but not for the printer.

In October 1666, Newton gathered his discoveries on the calculus of fluxions in a well-written small treatise (see chapter 1). In September 1668, Mercator's *Logarithmotechnia* was published. Here Newton could find results on infinite series that he had achieved a few years before. Mercator had gone very close, too close, to the binomial series. Newton was worried by the challenge posed by Mercator and composed *De Analysi*. Barrow, who was aware of some of Newton's discoveries, communicated *De Analysi* to Collins in July 1669.[3] This was the move to be taken in order to promote his young protégé. Collins was a philomath who enthusiastically gave English mathematicians publicity. He did so by supervising the publication of mathematical works and by favoring the correspondence between English, Scottish, and Continental numerati. Newton kept in touch with Collins until the mid-1670s. The latter was even able to put Newton to work on a revised and expanded edition of a treatise on algebra by the Dutchman Gerard Kinckhuysen (§4.1).[4]

In the early 1670s, Collins and Barrow proposed several editorial projects to Newton, most notably *De Analysi* could have been published as an appendix to Barrow's *Lectiones Geometricae* (1670).[5] Newton showed interest in these editorial

[2] Adams, Liveing, Luard, and Stokes, *A Catalogue of the Portsmouth Collection* (1888), pp. xix–xx, pp. 25–31. See Iliffe, "A Connected System" (1998).

[3] *Correspondence*, 1, p. 14.

[4] The translation from Dutch into Latin was the work of Mercator.

[5] Collins to J. Gregory (February 12/22, 1669/70): "I believe Mr Newton ... will give way to have it printed with Mr Barrows Lectures." *Correspondence*, 1, p. 26. See Whiteside's commentary

projects, but he soon changed his mind and his tract appeared only in 1711, when the priority dispute with Leibniz was at its peak, in a collection of Newtonian mathematical works edited by William Jones (§17.1.4).

The fate of *De Analysi* is a typical Newtonian story. The next major treatise, *De Methodis* (see chapter 8)—perhaps Newton's masterpiece on series and fluxions—suffered an even worse destiny, being printed posthumously in an English translation in 1736.[6] Newton composed *De Methodis* in 1670–1671, in a moment of mathematical creativity that almost obscures the *anni mirabiles*. In *De Methodis* one can find the most advanced Newtonian techniques of series expansion, an exposition of the basic concepts and rules of the fluxional method, the application of the direct method of fluxion to the calculation of tangents and curvatures, and two systematic catalogues of curves that bring the squaring of curves to perfection. Projects of printing *De Analysi*, *De Methodis*, or Newton's comments on Kinckhuysen's algebra were repeatedly considered.[7] However, *De Methodis* did not go to press during Newton's lifetime, and the notes on Kinckhuysen were later integrated within the body of *Arithmetica Universalis* (1707). As a frustrated Collins wrote to James Gregory in June 1675, "Mr Newton intends not to publish anything, as he affirmed to me, but intends to give in his lectures yearly to the publick library."[8] In this case Newton's lectures could be transcribed.[9] But even this less ambitious project remained still born, as Collins learned from Newton in September 1676: "[T]hough about 5 years agoe I wrote a discourse in wch I explained ye doctrine of infinite aequations, yet I have not hitherto read it but keep it by me."[10]

Much to Collins's frustration, Newton expressed his reluctance to print his mathematics. Newton was quite determined in not allowing his mathematical jewels to escape from his hands. To the lucky few who had corresponded with him on mathematical subjects and who had had access to his manuscripts he ordered silence and secrecy. In October 1676 he wrote to Henry Oldenburg, the secretary of the Royal Society who, after Collins, enjoyed Newton's overtures on mathe-

---

in MP, 2, p. 168. Barrow's geometrical lectures were reprinted in 1674 together with the optical lectures.

[6] Newton, *The Method of Fluxions and Infinite Series* (1736).

[7] See, for instance, Newton to Collins (May 25, 1672): "I may possibly complete the discourse of resolving Problemes by infinite series of wch I wrote the better half ye last christmas wth intension that it should accompany my Lectures [*Optical Lectures*], but it proves larger than I expected & is not yet finished." *Correspondence*, 1, p. 161.

[8] Collins to J. Gregory (June 29, 1675) in Hiscock, *David Gregory, Isaac Newton and Their Circle* (1937), p. 310.

[9] Collins to J. Gregory (December 24, 1670): "Mr Barrow told me the Mathematick Lecturer there is obliged either to print or put 9 Lectures yearly in Manuscript into the publick Library, whence Coppies of them might be transcribed." *Correspondence*, 1, p. 54.

[10] Newton to Collins (September 5, 1676) in *Correspondence*, 2, p. 95.

matics, "Pray let none of my mathematical papers be printed wthout my special licence."[11]

16.1.2 Explanations

Several explanations of Newton's secretive attitude have been given. Some historians refer to the cost of book printing after the Great Fire in 1666. Some describe Newton as an odd, sometimes even neurotic character, who isolated himself in an ivory tower. Some describe the aftermath of the dispute on optics as a cause for Newton's reluctance to publish. Some think that in the 1670s Newton's interest shifted from mathematics to other subjects (primarily alchemy, theology, and history); he would simply have lost a motivation to rework his mathematical manuscripts for the press. There is a grain of truth in each of these explanations.

It is true that in the 1670s the booksellers in London were in crisis. Whiteside has observed that the economic failure of Wallis's *Mechanica* (1670), Horrocks's *Opera Posthuma* (1673), and Barrow's *Lectiones Opticae & Geometricae* (1674) did not help those who wished to enter the market of mathematical books.[12] However, the printing of mathematical books did not come to a halt. Barrow continued in the 1670s to publish his lectures as well as his editions of Archimedes, Theodosius, and Apollonius. To take some further examples, in the 1670s, John Crooke printed Hobbes's pamphlets on the circle quadrature, hardly to be defined as best sellers, and William Godbid printed the two volumes of Kersey's *Algebra* (1673–1674). The latter saw the light thanks to the encouragement of Collins, who would have certainly loved to be the midwife of one of Newton's tracts.[13]

The fact that Newton was acutely sensitive to criticism is well documented. Further, the effect of the dispute on the *experimentum crucis* cannot be overestimated. His great paper of 1672 was fiercely attacked, and this frustrating experience was to drive Newton away, maybe with revenge, from publishing his results in other fields of enquiry.[14] Tired after years of polemic, he wrote to Hooke in 1676,

> There is nothing wch I desire to avoyde in matters of Philosophy more then contention, nor any kind of contention more then one in print.[15]

---

[11] Newton to Oldenburg (October 26, 1676) in *Correspondence*, 2, p. 163.

[12] Whiteside documented Collins's apprehensions in MP, 3, pp. 5–6.

[13] On Collins's role in promoting English algebra, see Pycior, *Symbols, Impossible Numbers, and Geometric Entanglements* (1997), pp. 70–102. On Hobbes's tracts, see Jesseph, *Squaring the Circle* (1999).

[14] In a letter dated May 25, 1672, concerning the project of printing his lectures on optics, Newton wrote to Collins: "I have now determined otherwise of them; finding already by that little use I have made of the Presse, that I shall not enjoy my former serene liberty till I have done with it." *Correspondence*, 1, p. 161.

[15] February 5/15, 1675/76, *Correspondence*, 1, p. 416.

He also divulged the idea that he was not interested in mathematics any longer. As a worried Collins explained in 1675 to James Gregory, "[B]oth he and Dr Barrow [are] beginning to thinke mathcall Speculations to grow at least nice and dry, if not somewhat barren." Newton, he added, was "intent upon Chimicall Studies."[16]

But Newton, notwithstanding his reluctance to go to press and his declarations of disinterest toward mathematics and natural philosophy, was still active as a mathematician. The fourth, fifth, sixth, and seventh volumes of *Mathematical Papers* edited by Whiteside reveal that Newton worked on mathematics until 1696 without interruptions. In the 1670s and early 1680s he produced his lectures on algebra (printed as *Arithmetica Universalis* in 1707). He also devoted efforts to the classification of cubics (he further improved this topic in 1695, producing the treatise printed in 1704 as *Enumeratio Linearum Tertii Ordinis*); to the differential method (*Methodus Differentialis*, printed in Jones's collection of Newton's mathematical tracts); and to geometry. In the early 1690s he concerned himself with squaring techniques, writing several versions of what became *Tractatus de Quadratura Curvarum*, printed as an appendix to the *Opticks* in 1704.[17] However, it was only at the beginning of the eighteenth century that Newton chose to print his mathematics.

16.1.3 A Method "Unfit to Commit to Posterity"

To the preceding explanations of Newton's rejection of print publication of his new analysis of series and fluxions, I would like to add another one, which, in my opinion, has been overlooked. The interpretation I propose is that in the 1670s, indeed as early as *Optical Lectures* (1670), Newton showed a marked concern about the role that mathematics could play in a broad-ranging and ambitious program in natural philosophy. He was convinced that geometry could inject certainty into natural philosophy, making it possible to surpass the probabilism inherent in both the Cartesian hypothetic-deductive method and in Baconian inductivism (see chapter 2). The new analysis that Newton practiced so well, and that aroused the interest of Collins, was not "worthy of public utterance," Newton wrote in 1671.[18] In 1694, Gregory noted that according to Newton, "[O]ur specious algebra is fit enough to find out, but entirely unfit to consign to writing and commit to posterity."[19] New-

[16] Collins to J. Gregory (October 19, 1675) in *Correspondence*, 1, p. 356.
[17] For bibliographical details, see A Brief Chronology of Newton's Mathematical Work, following chapter 18.
[18] MP, 3, p. 279.
[19] David Gregory's memorandum of a May 1694 visit to Newton. "Algebram nostram speciosam esse ad inveniendum aptam satis at literis posterisque consignandum prorsus ineptam." Edinburgh University Library, MS Gregory C42. Translation by Whiteside in MP, 7, p. 196. See also *Correspondence*, 3, p. 385.

ton therefore began searching for a synthetic, "more perspicuous and resplendent," method (1671) (§9.1).[20]

The *experimentum crucis* polemic that began in 1672 displeased Newton, since he was drawn into litigation in a field that he had hoped left no room for controversy. His results, based on mathematical deduction from phenomena, were, in his opinion, indisputable. Newton was well aware that the publication of his method of series and fluxions would have involved him in an even more embarrassing polemic. He knew what kind of criticisms Wallis had to face against Fermat and Hobbes (§8.2). These criticisms would have been lethal for Newton for two very solid reasons. First, he would have had to defend his methods by claiming that they were analytical methods of discovery, as such to be valued not in function of their rigor but in function of their heuristic power. This move, which was Wallis's reply, was not available to Newton, since he defined himself as a philosopher who could achieve certainty by certain geometrical means. Second, in printing his analytical method Newton would have had to align himself with a genre of mathematical literature practiced by moderns who often referred to Descartes as their master.

Newton in the 1670s developed a marked anti-Cartesian position (see part II). In his notes on Kinckhuysen, and in his works on Pappus and on the restoration of Euclid's *Porisms*, he distanced himself from the Cartesian canon trying to devise alternative analysis and synthesis. He very often contrasted the ancients to the moderns and sided passionately with the first. The ancient method, in his opinion, was "more elegant by far than the Cartesian one," which he deemed so tedious as to provoke nausea. Nothing written in a style different from the ancient one was worthy to be read (late 1670s).[21] One could use equations in the heuristic analytical stage, but when demonstrating a proposition equations had to be neglected. The ancients, Newton repeated, "never introduced arithmetical terms into geometry; while recent people, by confusing both, have lost the simplicity in which all elegance of geometry consist" (late 1670s).[22]

A model of geometrical style landed on Newton's table in 1673, when he received a complimentary copy of Huygens's *Horologium Oscillatorium.* Huygens showed how cutting-edge natural philosophy could be carried on in synthetic geometrical style. The *Horologium* made a deep impression on Newton, who never ceased to praise Huygens as the restorer of ancient mathematical tradition. Henry Pemberton, editor of the third edition of the *Principia* (1726), and a privileged witness of Newton's last years, wrote in *View of Sir Isaac Newton's Philosophy* (1728),

> I have often heard him censure the handling of geometrical subjects by algebraic calculations; ... he frequently praised Slusius, Barrow and Huygens for not being

[20] MP, 3, pp. 283, 331.
[21] MP, 4, p. 277.
[22] MP, 5, p. 429.

> influenced by the false taste, which then began to prevail. He used to commend the laudable attempt of Hugo de Omerique to restore the ancient analysis, and very much esteemed Apollonius' book *De Sectione Rationis* for giving us a clearer notion of that analysis that we had before ... Sir Isaac Newton has several times particularly recommended to me Huygens' style and manner. He thought him the most elegant of any mathematical writer of modern times, and the most just imitator of the ancients. Of their taste and form of demonstration Sir Isaac always professed himself a great admirer: I have heard him even censure himself for not following them yet more closely than he did; and speak with regret of his mistake at the beginning of his mathematical studies, in applying himself to the works of Des Cartes and other algebraic writers before he had considered the elements of Euclide with that attention, which so excellent a writer deserves.[23]

In these circumstances Newton shied away from printed publication of his new analysis. The synthetic version of his method of fluxions (§9.2), which he elaborated in "Geometria Curvilinea" (ca. 1680), was eventually printed in the *Principia*, but Newton resisted the project of printing his new analysis. Very much as the ancients had done, according to Newton's historical reconstruction, he concealed the analysis. Somewhat parenthetically, I would like to surmise that Newton's mathematical classicism and criticism of Cartesian mathematical method might be in resonance with his agendas in cosmology, chronology, and theology, where similar concerns about Cartesianism and the ancients emerge.

Finally, I would like once more to underline two points:

1. Newton's new analysis took the form of a series of algorithmic rules that could be displayed by application to particular examples. This is particularly evident in Newton's handling of series, for instance, the technique of resolution of affected equations (§7.5) and quadratures (§8.4). Rather than a theory that could be presented in a book, the new analysis was a formidable panoply of techniques whose handling could be best explained and communicated orally or via correspondence. These Newtonian techniques, when translated into a more algebraic and general language that was developed later in the eighteenth century (e.g., by using symbols for functions, summations, etc.), acquire a generality and recursive character that is lacking in Newton's original presentation. The importance of personal contact in trading skills in the new analysis cannot be overestimated, since the printed page was not a totally reliable support for communicating the new analysis techniques. Indeed, as Newton learned from the annoying dispute on the *experimentum crucis*, the printed page could also fail to reliably convey the craft of experimental techniques. Newton admitted to his rooms a number of mathematicians (Craig, Halley, Gregory, Fatio) who could be instructed by the master. Similarly, the Leibnizian calculus spread on

[23] Pemberton, *View of Sir Isaac Newton's Philosophy* (1728): *Preface.*

the Continent via extensive correspondence among its practitioners, or via such means as *privatissima* imparted by Johann Bernoulli to L'Hospital in Paris, or to young Euler in Basel. The craft had still to be learned in a *bottega del maestro* rather than by reading a textbook. This condition changed rapidly as the production of textbooks and journals flourished at the beginning of the eighteenth century.

2. For Newton, as for many of his contemporaries, it was geometry, not algebra, that delivered generality. We are accustomed, especially after the development of algebra in the nineteenth century, to consider geometry a particular model of a more general algebraic theory. This conception is not applicable to the late seventeenth century. For mathematicians such as Huygens, Barrow, and Newton, geometry provided a much more general language than algebra. Algebra could cover only a sector of the objects treated by these mathematicians, a limitation that Descartes internalized in his *Géométrie* by rejecting treatment of mechanical curves (§3.2.3). This is particularly evident for trigonometric magnitudes. For instance, in Newton's time there was no notation for hyperbolic functions, which Newton represented geometrically by referring to conic areas, for instance, the construction of Corollary 3, Proposition 41, Book 1, of the *Principia* (§12.2.2).[24] He did not have the notation for many transcendental functions. The preference given to printing geometry rather than the new algebraic analysis is not so surprising if one takes into consideration the limited range of application of algebra compared to geometry available to late-seventeenth-century mathematicians.

16.2 Manuscript Circulation

Newton had means other than printing to let the outside world know that he was a great mathematician and to acquaint the cognoscenti with the new analysis that flourished in his mathematical manuscripts. In the period preceding the printing of the *Principia*, most of Newton's mathematical discoveries were rendered available to the mathematical community through rather oblique ways. Newton engineered a complex publication strategy. He allowed some of his mathematical discoveries to be divulged through letters (§16.3) and manuscript circulation. Manuscripts were shown to a selected group of experts in the field (such as John Collins, John Craig, Edmond Halley, John Flamsteed, David Gregory, and Nicolas Fatio de Duillier), who visited Newton in Cambridge. They were deposited at the Royal Society in London or as Lucasian Lectures in the University Library at Cambridge, and they were even copied (sometimes in mutilated form).

[24] One of the first occurrences of the notation for hyperbolic functions is in Riccati and Saladini, *Institutiones Analyticae* (1765–1767), 2, p. 152.

As mentioned (§16.1.3), Newton's publication strategy was related to the values that he endorsed and that he wished to defend. From the 1670s, Newton realized that the standards of validation that he aimed at were often above his mathematical practice, a practice he was led to downgrade to the level of a heuristic technique not worthy of printed publication.

Recall that *De Analysi* was sent by Barrow to Collins in July 1669.[25] Collins not only informed a number of mathematicians of Newton's researches in the new analysis,[26] but also made at least two transcriptions. Collins returned the original to Barrow but kept a copy himself. In 1677 he made a secondary transcript, which was sent to Wallis and, after Wallis's death, passed into the hands of David Gregory.[27] Leibniz was allowed to read and transcribe parts of the first transcript during his visit to London in 1676.

De Methodis enjoyed a similar publication story. John Colson's English translation of it, which appeared in 1736, is based on a transcript made by William Jones about 1710.[28] Whiteside mentioned a secondary transcript, now lost, from Jones's copy made by James Wilson about 1720. Samuel Horsley used both of these transcripts in his edition of Newton's *Opera*.[29] In early 1685 the Scottish mathematician John Craig was allowed to inspect several manuscripts by Newton and probably made copies of parts of *De Methodis*,[30] and Thomas Pellet received an incomplete copy from Jones.[31] In volume 3 of *Mathematical Papers*, Whiteside reproduced a "Tractatus de Seriebus Infinitis et Convergentibus" in David Gregory's hand whose first folios are an abridged transcript of the opening sections of *De Methodis*.[32] The last two folios are Gregory's jottings (in English) taken from Newton's notes on Kinckhuysen and from the two *epistolae* to Leibniz. A letter from Craig to Colin Campbell, dated January 30, 1688, on a "general method for Finding the Curvature ... copied out of Mr Newtons manuscript" seems to indicate that Craig must have seen either Gregory's transcript or a copy of it, since in it he deals

[25] Barrow to Collins (July 31, 1669) in *Correspondence*, 1, p. 14.

[26] Among others, William Brouncker, James Gregory, Rene-François de Sluse, and Giovanni Alfonso Borelli. See the evidence discussed in MP, 2, p. 168.

[27] William Jones found Collins's copy in the papers of Collins that he had acquired, and used this copy and the original that Newton lent him in 1709 in order to produce his edition. See Newton *Analysis per Quantitatum* (1711), Praefatio [n.p.]. Newton's original is now, bound with Jones's first transcript, at the Royal Society Library (MS LXXXI, No. 2), and the secondary transcript is in St. Andrews University Library (MS QA 33 G8 D3, ff. 1–10). See Whiteside's commentary in MP, 2, pp. 206–7, note 2.

[28] Macclesfield Collection (Cambridge University Library), Add. 9597.9.2. A transcript in Jones's hand of Colson's commentary is in the Macclesfield Collection as Add. 9597.9.21.

[29] Newton, *Opera* (1779–85), 1, p. 390. See MP, 3, p. 32.

[30] MP, 7, pp. 3–4.

[31] Robins, *Mathematical Tracts* (1761), 2, pp. 357–8, and MP, 3, p. 11n.

[32] Edinburgh University Library, Gregory MSS A56, in MP, 3, pp. 354–72.

with the problem of curvature in terms that show strong similarities with Gregory's transcript rather than with Newton's 1671 tract.[33]

As far as the scribal dissemination of Newton's fluxional writings one should note here that there is evidence that the October 1666 Tract on Fluxions, one of the earliest systematizations of fluxions achieved by Newton, enjoyed a limited circulation. Copies can be found both in the Portsmouth and in the Macclesfield collection. These copies must be late ones, certainly post 1690, since the dot notation for velocities is used.[34]

There is a fascinating and complex story about the circulation of manuscripts related to the *Principia*. As I have shown in *Reading the Principia* (1999), knowledge about the quadratures necessary to complete the gaps of several demonstrations (especially in the Scholium to Proposition 35/34, Book 2, on the solid of least resistance and in Proposition 41, Book 1, on central force motion) was shared by Newton's disciples (see figure 12.3).[35]

From this regrettably incomplete information about the dissemination of Newton's mathematical manuscripts one understands that Newton did not close himself in an ivory tower. He had a publication strategy for his mathematical discoveries that can be best defined as scribal publication. As Love has shown, the practice of scribal publication flourished in Restoration England. Love described the practice of publishing texts in handwritten copies within a culture that had developed so-

[33] Craig to Campbell (January 30, 1688) in *Correspondence*, 3, pp. 8–9. See Whiteside's discussion in MP, 3, p. 354n.

[34] According to Whiteside, "[T]here exists, in private possession, an early contemporary copy in the hand of Newton's room-mate and amanuensis, Wickins, which was possibly destined for John Collins though we have no evidence to show that it ever passed out of Newton's possession before his death. With this copy are some extra sheets of notes in the hand of William Jones [and especially on Prob. 9]: we may conjecture that these are first drafts for the rearranged copy in his hand in the University Library, Cambridge [Add. 3960.1, ff. 1–50]. An eighteenth-century entry, in the hand of James Wilson, affirms correctly of the latter that 'The Transcriber has here put $\dot{x}$, $\dot{y}$ and $\dot{z}$ for p, q and r of the Original ... Here seems to be some transpositions and interpolations, as Mr Jones was wont to make in those papers of Sr Isaac Newton, which he distributed to his scholars, that none might make a perfect book out of them'." MP, 1, p. 400. Wilson prints extracts of Wickins's copy then in the hands of Newton's executor Pellet. Robins, *Mathematical Tracts* (1761), 2, pp. 351–6. Another copy (in Jones's hand?) of the October 1666 Tract on Fluxions is in the Macclesfield Collection, Add. 9597.9.1. In this copy, too, the post-1690 dotted notation is changed in place of Newton's early notation.

[35] Newton's fluxional solution of the problem of the solid of least resistance was circulated. In 1694, David Gregory got the solution from Newton: John Keill and William Jones transcribed it. Newton's fluxional solution was published by Fatio de Duillier in *Lineae Brevissimi Descensus Investigatio Geometrica* (1699); by Charles Hayes in *A Treatise of Fluxions* (1704), on pp. 146–50; and in 1729 in an Appendix to Motte's English translation of the *Principia*, on pp. 657–9. Newton's manuscripts relating to the solid of least resistance were found in the Portsmouth papers and published by Adams, Liveing, Luard, and Stokes in *A Catalogue of the Portsmouth Collection* (1888). See also *Correspondence*, 3, pp. 323, 375–7, 380–2.

phisticated means of generating, transmitting, and even selling such copies. Love studied the ways in which manuscripts of political, literary, and musical content circulated in Restoration England. The invention of printing did not, of course, obliterate the practice of manuscript circulation. However, after the invention of printing, scribal publication was pursued with specific purpose. As Love remarked,

> [T]here is a significant difference between the kinds of community formed by the exchange of manuscripts and those formed around identification with a text. The most important is that the printed text, being available as an article of commerce, had no easy way of excluding readers. Interesting in the choice of scribal publication . . . was the idea that the power to be gained from the text was dependent upon possession of it being denied to others. . . . Print publication implied the opposite view of a community being formed by the public sharing of knowledge.[36]

Newton tried to keep control over the dissemination of his mathematical manuscripts. One learns something about the extent and modality of such practice from James Wilson, who wrote to Newton on December 15, 1720,

> I saw the other day in the hands of a certain person, several Mathematical Papers, which, he told me, were transcribed from your Manuscripts. They chiefly related to the Doctrine of Series and Fluxions, and seemed to be taken out of the Treatises you wrote on those subjects in the years 1666 and 1671. . . . These papers, I observed, had been very incorrectly copied, so that I endeavoured all I could, to dissuade the Possessour of them from getting them printed, of which nevertheless he seemed very fond. . . . I have since met with another Person, who told me, he had likewise a Copy of your Manuscripts. But he would not let me see them, or inform me how he came by them. I imagine, when you sent any of your Friends your papers, the person they got to transcribe them, took a double copy, which is a frequent practice, in order to make profit by it. So that they are in different hands.[37]

By the 1720s, when Newton was a great celebrity, the circulation of unauthorized copies of his mathematical manuscripts was, it seems, not uncommon. Newton's manuscripts had been circulating for too long and they had fallen into too many hands. It is not known, however, how reliable is the recollection according to which Halley and Raphson, when they examined the original manuscript of *De Methodis*

[36] Love, *Scribal Publication* (1993), pp. 183–4. Another important study on the production and circulation of manuscripts in seventeenth-century England is Beal, *In Praise of Scribes* (1998). While Beal concentrates more on the first half of the century, what he has to say about the status of scribes, and the nature of scriptoria, can be—with some caution—extended to the Restoration and perhaps even to the early eighteenth century. Most notably Beal converges with Love in characterizing the audience, or coterie, created by manuscript circulation as consciously seeking selectiveness and an awareness to be above the common level of the market place.

[37] Keynes MS 143.1 (King's College, Cambridge). James Wilson to Newton (December 15, 1720) in *Correspondence*, 7, pp. 107, 109.

in Cambridge in 1691, found it “very much worn by having been lent out.”[38] The taking of copies and double copies was a frequent practice. Love pointed out that booksellers often employed professional copyists in order to produce transcripts for sale. From Collins’s days of pure enthusiasm for English mathematics the economic interest in having access to Newton’s mathematical manuscripts had superseded the scientific curiosity. The most devout Newtonians began to adopt the practice of mutilating these copies in order to avoid pirated printing. James Wilson remarked that Jones, the man who possessed a large corpus of Newtonian manuscripts, “was wont to curtail or otherwise disguise the [Newton’s] papers, he communicated to his scholars, that none might make out a compleat book.”[39] In this case, Jones, a mathematics teacher, made use of the Newtonian materials he held in order to acquire prestige and students, but did not want to dissipate his treasure by letting the manuscripts reach the hands of other parties in their entirety.

It is difficult to establish who had access to, and what was known about, Newton’s mathematical manuscripts. There was an inner circle of acolytes who could have had access to Newton’s papers by visiting him in Cambridge or corresponding with him. This was the case with Collins, who received just some hints about Newton’s fluxions but was well aware of Newton’s researches on series, algebra, and the organic description of curves.[40] In 1674, John Flamsteed was given a set of notes on algebra by Newton.[41] The lectures on algebra were deposited in the University Library in 1684, and in principle they became public. The Scots Craig and David Gregory were able to transcribe some of the mathematical manuscripts after their visits to Cambridge in 1685 and 1694, respectively. Gregory summarized what he saw in Cambridge in a treatise on fluxions now in Christ Church (Oxford).[42] Gregory’s short treatise itself circulated; a copy (in the hand of Colson?) and a scribal copy are preserved in the Macclesfield Collection.[43] Halley, Raphson, and Fatio were certainly allowed by Newton to read his mathematical manuscripts.

---

38 This statement can be found in Raphson, *The History of Fluxions* (1715), pp. 2–3, and therefore cannot be accounted as completely reliable. Raphson was defending Newton in the priority dispute, and any proof of circulation of knowledge about fluxions was instrumental to Newton’s cause. See MP, 3, p. 32n.

39 Robins, *Mathematical Tracts*, 2, pp. 357–8. See MP, 3, p. 11n.

40 See, for instance, Newton to Collins (August 20, 1672) in *Correspondence*, 1, pp. 229–32.

41 This manuscript was first edited by Edleston, who reproduced an original paper in Newton’s hand pasted at the beginning of vol. 42 of Flamsteed’s manuscripts at Greenwich. “At the bottom are the words ‘Mr. Newton’s paper given at one of his lectures, Midsummer, 1674.’ Flamsteed was at Cambridge from the end of May to July 13, and visited Newton.” J. Edleston, *Correspondence of Sir Isaac Newton and Professor Cotes* (1850), pp. 252–3. See Whiteside’s edition in MP, 5, pp. 32–3.

42 “Isaaci Newtoni Methodus Fluxionum Ubi Calculus Differentialis Leibnitii, et Methodus Tangentium Barowij Explicantur, et Exemplis Quamplurimis Omnis Generis Illustrantur.”

43 Macclesfield Collection (Cambridge University Library), Add. 9597.9.3 and Add. 9597.9.4.

For instance, in 1695, Halley was given *De Quadratura* for transcription.[44] Another manuscript that was circulated is the short treatise on the construction of equations[45] that Newton wrote in the early 1670s. It was sent on August 20, 1672, to Collins, who described its content to Wallis[46] and made a copy that passed into the hands of William Jones, who probably contemplated the possibility of publishing it.[47] In the late 1690s the manuscript of *Enumeratio Linearum Tertii Ordinis* was shown to David Gregory.[48] There was also a larger circle of philomaths, whose composition is difficult to determine, who were informed, sometimes in mutilated form, by the acolytes.

16.3 Correspondence with Collins and Leibniz

16.3.1 Collins

In Newton's time, notwithstanding the establishment of scientific journals in the second half of the seventeenth century, correspondence was still an important vehicle for exchanging information on cutting-edge research. One of the most active in promoting correspondence between mathematicians was John Collins, a mathematical entrepreneur who from 1667 was employed as librarian for the Royal Society. A clerk who worked most of his life as a government accountant, but who had previous experience with bookbinding and navigation, Collins established himself as a mathematics teacher, prolific book editor, and liaison between Scottish, English, and continental geometers. It is to him that Barrow turned in order to promote his young protégé, sending a copy of *De Analysi* (§1.5).

Newton engaged in a rich mathematical correspondence with Collins during the period 1669–1674. Collins transcribed Newton's letters and manuscripts, circulated knowledge about them, and made Newton aware of advances achieved by other mathematicians.[49] After his death in 1683 his papers and library passed into William Jones's hands; it is to this archive that Newton turned when, during the priority dispute with Leibniz, he looked for documentary evidence of his early advances in mathematics.

---

[44] Halley to Newton (September 7, 1695): "I have not yett returned your Quadratures of Curves, having not yet transcribed them, but no one has seen them, nor shall, but by your directions; and in a few days I will send you them." *Correspondence*, 4, p. 165.

[45] Add.3963.9, ff. 70r–106v, in MP, 2, pp. 450–517.

[46] Newton to Collins (August 20, 1672) in *Correspondence*, 1, p. 231, and Collins to Wallis (?1677/8) in *Correspondence*, 2, p. 243.

[47] See Whiteside's commentary in MP, 2, pp. 450–1.

[48] David Gregory noted in July (?) 1698: "sunt 16 genera Curvarum secundi generis, et 76 Curvae Newtonus conscripsit tractatum de illis quem mihi impertietur ut eum edam." *Correspondence*, 4, p. 277.

[49] Pycior, *Symbols, Impossible Numbers, and Geometric Entanglements* (1997), pp. 70–87.

Collins was mainly interested in algebra. He busied himself with the project of producing an updated textbook on algebra available to interested readers in Britain. He is the midwife of both Pell's edition of Rahn's algebra and Kersey's *Elements*.[50] With Newton he embarked on an ambitious editorial project, the edition of Mercator's Latin translation of Kinckhuysen's Dutch textbook on algebra. Newton's notes on Kinckhuysen were the first steps toward a research program that eventually led to *Arithmetica Universalis* (§4.1). Consequently, Newton's correspondence with Collins relates mostly to algebraic matters.

Of course, series entered prominently into the mathematical discussion between Newton and Collins, since series were conceived of as a topic belonging to algebra. Collins knew everything about Newton's quadrature methods via infinite series as developed in *De Analysi*, and he discussed Newton's results with Brouncker at the Royal Society and with James Gregory. Further, Collins informed Newton about Gregory's exceptional results in this field. It is thanks to this exchange that Newton learned that all the series for trigonometric magnitudes that he had displayed in *De Analysi* (§7.4) were accessible to the Scot as well. Newton's infinite series were therefore in the public domain by the mid-1670s via scribal circulation and correspondence.

But Newton communicated to Collins more than algebra. There are letters concerning the organic description of conics, the calculation of logarithms, and—exceptionally—the method of fluxions. In the mid-1670s, Collins thus became the person who was best informed about Newton's mathematical work. A report he compiled for Wallis in 1677/1678 showed that he was well aware of the notes on Kinckhuysen and of the work on series (namely, *De Analysi*) as well as of some of Newton's work on geometry, algebra, and optics.[51]

16.3.2 The Tangent-Letter to Collins, December 10, 1672

What did Collins know about the method of fluxions, direct (§8.3) and inverse (§8.4), as developed by Newton in *De Methodis*? It seems very little. Whenever Collins was asked to report about Newton's discoveries he referred to algebra, to the organic description of conics, and to the quadrature techniques via series expansion of *De Analysi* (§7.4).

---

[50] Rahn, *An Introduction to Algebra* (1668); Kersey, *The Elements of That Mathematical Art Commonly Called Algebra* (1673–4).

[51] In a letter, written presumably in 1677/78, he informed Wallis about the extent of Newton's work. Wallis was working at his English *Algebra*, which eventually appeared in 1685, a work providing a historical presentation of the development of algebra. Collins listed the following: (i) an introductory part from Kinckhuysen, (ii) a discourse about bringing problems to an equation, (iii) a treatise about the construction of problems and equations, (iv) a discourse concerning several kinds of infinite series, (v) a treatise *de locis*, (vi) "the same applyed to Dioptriques." *Correspondence*, 2, pp. 242–3.

In a famous letter dated December 10, 1672 (the tangent-letter), Newton hinted at his method for drawing tangents, but he illustrated it with a simple cubic curve.[52] This letter originated from the fact that René-François de Sluse, after receiving a copy of Barrow's *Lectiones Geometricae*, had replied that he had a tangent method, too, and had presented it to *Philosophical Transactions*.[53] Newton replied to Collins that he was "heartily glad" to find that the "forreign Mathematicians ... are falln into the same method of drawing Tangents wth me." He gave an example ($x^3 - 2x^2y + bx^2 - b^2x + by^2 - y^3 = 0$) fully within Sluse's power but added that his "Generall Method," contrary to Hudde's and Sluse's (which he defined as corollaries of his) extended not only to geometrical lines but also to equations not "free from surd quantities" (§8.3.3) and even to mechanical curves (§8.3.4). His method, Newton continued, could be used for the calculation of "crookedness, areas, lengths, centers of gravity of curves."[54] Collins transcribed this letter and sent it to James Gregory.[55] Newton gave no details, however, about his general, direct method of fluxions. The premises for a controversy between Newton and Sluse were silenced, it seems, at a council of the Royal Society. Newton's apologetic concession was transmitted to Sluse via Oldenburg.[56]

16.3.3 The Quadrature-Letter to Collins, November 8, 1676

The same secretive attitude can be discerned with regard to the inverse method. When, in his letter dated August 17/27, 1676, after receiving the *epistola prior*, Leibniz informed Newton about his transmutation method for squaring curves applied to the circle quadrature,[57] Newton wrote to Collins (on November 8, 1676) that he had a much more powerful technique:

> I say there is no such curve line but I can in less then half a quarter of an hower [sic] tell whether it may be squared or what are ye simplest figures it may be compared wth, be those figures Conic sections or others.[58]

The attentive reader of Newton's mathematical papers will not miss what Newton was hiding here. "Curves that may be squared" and "curves that may be compared

[52] *Correspondence*, 1, pp. 247–52 (on p. 247).
[53] On Barrow's tangent method, see §8.1.4. For Sluse, see *Philosophical Transactions*, 7 (1672–3), pp. 5143–7, and proof in 8 (1673), p. 609.
[54] *Correspondence*, 1, pp. 247–8.
[55] Collins to James Gregory (February 20, 1673) in J. Gregory, *Tercentenary Memorial Volume* (1939), p. 258. There is also a transcript in tremulous imitation of Newton's hand: see *Correspondence*, 2, p. 14.
[56] Birch, *History of the Royal Society* (1756–1757), 3, p. 92. Newton to Oldenburg (June 23, 1973) in *Correspondence*, 1, p. 294. Hofmann, *Leibniz in Paris* (1974), pp. 263–4. See also Collins to Newton (June 18, 1673) in *Correspondence*, 1, pp. 288–9.
[57] *Correspondence*, 2, pp. 57–64
[58] *Correspondence*, 2, p. 179.

with conic sections" are technical expressions denoting the two quadrature techniques (Method 2 and Method 3) that allowed Newton to write the two catalogues of *De Methodis* (§8.4.3, §8.4.4).[59] But after this tantalizing hint at his method of quadratures via the two catalogues, which were eventually printed at the end of *De Quadratura*, Newton added,

> This may seem a bold assertion ... but it's plain to me by ye fountain I draw it from, though I will not undertake to prove it to others.[60]

This letter to Collins (the quadrature-letter), composed in the context of the 1676 correspondence with Leibniz, is revealing of Newton's policy of publication. The fountain he wanted to hide comprised the two methods (Method 2 and Method 3) for squaring curves based on (in Leibnizian terms) anti-differentiation (first catalogue) and integration by variable substitution (second catalogue). This was the fountain that Newton, at this juncture of his mathematical development, kept secret. By contrast, integration via power series (basically what one finds in *De Analysi*) was made accessible in scribal form.[61]

16.3.4 The *Epistola Prior* for Leibniz, June 13, 1676

Newton's policy of publication is evident also in what he chose to communicate and hide in his 1676 *epistolae* to Leibniz. In these letters he was quite open about his quadrature methods via infinite series (*De Analysi*) but much more secretive about his "short ways" to square curves (*De Methodis*, Method 2 and Method 3).

---

[59] For instance, in *De Methodis*, Newton wrote, "Hitherto we have exposed the quadrature of curves defined by less simple equations by the technique of reducing them to equations consisting of infinitely many simple terms. However, curves of this kind may sometimes be squared by means of finite equations also, or at least compared with other curves (such as conics) whose area may, after a fashion, be accepted as known. For this reason I have now decided to add the two following catalogues of theorems." MP, 3, p. 237. Here Newton first referred to a method for squaring curves via reduction to equations consisting of an infinite number of terms (infinite series) and then presented two other methods. The former refers to curves that can be squared in finite terms, the latter to curves that can be compared to curves whose area is assumed to be known, as conic sections.

[60] *Correspondence*, 2, p. 180.

[61] Both the tangent-letter (December 10, 1672) and the quadrature-letter (November 8, 1676) to Collins played a significant role in the controversy with Leibniz. Most notably, the 1672 tangent-letter was reproduced in *Commercium Epistolicum* (1713), pp. 29–30; an excerpt of the quadrature-letter was printed in Newton, *Analysis per Quantitatum* (1711), p. 38. The tangent-letter is important because Leibniz was able to consult it during his second visit to London in October 1676. It is clear, however, that Leibniz was already in possession of the differential method and that Newton's tangent-letter does not contain the direct algorithm of fluxions. Newton claimed that the quadrature-letter was proof that he had composed part of *De Quadratura* before 1676. See MP, 3, p. 19n.

Table 16.1 A Scheme of the *epistola prior*

Incipit	20
Binomial theorem stated	21
Nine examples of binomial expansion	21–23
Extraction of affected roots in numerical equations (Newton-Raphson)	23
Extraction of affected roots in literal equations	24
"Quadratures and mechanical lines can be expressed via infinite series"	24
Quadratures, as in *De Analysi*, explained by nine examples	25–29
Example 1: arcsin series	25
Example 2: sine and versed sine series	25
Example 3: angular section	25
Example 4: area of elliptic sector	25–26
Example 5: arc length of ellipse	26–27
Example 6: inverse of ellipse's arc	27
Example 7: area of hyperbola (exponential series)	27–28
Example 8: arc length and area of quadratrix	28
Example 9: volume of ellipsoid of revolution	28–29
"The limits of analysis are enlarged by infinite equations"	29
Numerical approximations for circle's and hyperbola's area and arc length	29–31
Explicit	31

Note: The numbers indicate pages in *Correspondence*, vol. 2.

When Newton was asked by Oldenburg to prepare for Leibniz a summary of his mathematical researches, he replied with a letter dated June 13, 1676, the *epistola prior*, whose main theme is curve-squaring via series expansion (table 16.1).[62] Actually, this is exactly what Leibniz had requested. Indeed, Leibniz had previously asked Oldenburg about British advances in mathematics and had received the impression that series constituted the main research area in Britain. In his first letter for Leibniz, Newton presented results on quadratures via infinite series in the style of *De Analysi*. He began by stating the binomial theorem (§7.3) and then displayed his method of root approximation (the Newton-Raphson algorithm) and the related method of resolution of affected equations (§7.5). He made clear that these methods of series expansion are useful in the calculation of areas of curvilinear surfaces, volumes of curvilinear solids, and centers of gravity. He displayed these applications of series by nine examples. Typically, Newton showed how the moment of a flowing quantity can be expressed as a power series, and then determined the fluent by squaring term-wise. This is the main quadrature technique of *De Analysi* (§7.4). Once a series was obtained, Newton often reversed it, for instance, in the *epistola prior*, when he obtained the series for the arcsine, he reversed it and obtaind the series for the sine. In brief, the *epistola prior* is a good introduction to

[62] *Correspondence*, 2, pp. 20-32. The exchange of letters between Newton and Leibniz has been analyzed in detail in Hofmann, *Leibniz in Paris* (1974), pp. 225–76.

the binomial theorem (stated, not explained) and to some results on quadratures via infinite series of *De Analysi*.

16.3.5 Leibniz's First Reply, August 17/27, 1676

Leibniz replied on August 17/27, 1676, presenting what he called a general doctrine of transformations, the method of transmutation. Basically, this method consists in a geometrical transformation of infinitesimal components of a curvilinear surface. Leibniz appled it to the calculation of π and to the expression of logarithm, exponential, and trigonometric magnitudes.[63] Further, he asked for details and demonstrations that were lacking in the *epistola prior*. He wished to know how Newton could (i) demonstrate the binomial theorem, (ii) resolve affected equations, and (iii) reverse series.

16.3.6 The *Epistola Posterior* for Leibniz, October 24, 1676

Whereas the *epistola prior* presented the main results of *De Analysi*, the *epistola posterior* explained the details of the methods of *De Analysi* and hinted at the results on higher quadratures of *De Methodis* while omitting explanations about them.[64]

In the *epistola posterior*, dated October 24, 1676, Newton gave details about his discovery of the binomial theorem, describing the route he had followed in 1664 from Wallis's *Arithmetica Infinitorum* (§7.3). He explained the gist of the inductive generalization that allowed him to reach this result. Newton also dealt at length with the analytical parallelogram whereby affected equations are resolved (§7.5) and gave examples of series reversion (§7.4). In doing so, he fully answered the questions Leibniz had posed in the letter dated August 17/27, 1676.

Newton did more than this. Probably because Leibniz's letter of August 17/27 had made him aware of the stature of the German, he wrote at length about most of his mathematical discoveries. Most notably, he hinted at all three methods for the quadrature of curves in *De Methodis* but was very secretive about them; he stated results but gave no instruction about how to achieve them. I suspect that the purpose of the *epistola posterior* was to impress Leibniz with the full breadth and depth of Newtonian mathematics. The *epistola posterior* is indeed a carefully written treatise. A significant selection of Newton's mathematical discoveries is included, but its author presented them with varying degrees of secrecy (table 16.2).

Newton revealed everything related to series and their use in the quadrature of curves. He also explained the resolution of affected equations via the analytical

[63] *Correspondence*, 2, pp. 57–64.
[64] *Correspondence*, 2, pp. 110–29.

Table 16.2 A Scheme of the *epistola posterior*

Incipit	110–111
Discovery and demonstration of binomial theorem explained	111–113
Application of binomial theorem to logarithm calculations	113–114
History of correspondence with Collins and James Gregory	114–115
A treatise [*De Methodis*] written in 1671	114–115
First anagram, this is the foundation for rendering quadratures simpler	115
Prime theorem (Method 2), no demonstration provided	115
Applications of prime theorem	115–117
Quadratures via reduction to conic areas (Method 3), no demonstration provided	117
Rectification of cissoid	117
Organic description (with no calculation) of cubics	119
Partial list of quadratures from the second catalogue of curves (Method 3)	119–120
Method 3 achieved "withdrawing from the contemplation of the figures"	120
Leibniz's series for π deduced and criticized as converging too slowly	120
Summation of numerical series	120–126
Analytical parallelogram used in resolution of affected equations explained	126-127
Reversion of series explained	127–129
Second anagram on two methods on the inverse method of tangents; the two methods are Method 1 and the method of undetermined coefficients	129
Explicit	129

Note: The numbers indicate pages in *Correspondence*, vol. 2.

parallelogram and the reversion of series. This is basically the content of *De Analysi* (chapter 7).

Newton also hinted at a treatise where "infinite series played no great part." This treatise (*De Methodis*) contained, among "few other things," the "method of drawing tangents." The foundation of this method was concealed, as was customary in the period, behind an anagram (the first of two occurring in the *epistola posterior*) whose meaning is: "given an equation involving any number of fluent quantities to find the fluxions, and conversely."[65] The decoding of this anagram would not, of course, have been of any help to Leibniz. The first anagram is a contraction of the basic Problem 1 and its inverse Problem 2 of *De Methodis* (§8.3, §8.4). Most probably, the role of the first anagram was to record the existence of *De Methodis* by quoting one of its crucial passages.[66]

[65] *Correspondence*, 2, p. 134.

[66] Recall that Problem 1 in the *De Methodis* is "Given the relation of the flowing quantities to one another, to determine the relation of the fluxions." MP, 3, p. 75. Problem 2 is "When an equation

It is generally believed that in this anagram Newton was hiding the calculus. This idea is both too general and inaccurate. There is no secret mathematical theorem hidden in the first anagram, whose decoding is really an anticlimax. What Newton's priorities were, what he was proud about, and what he wanted to hide are apparent from the general structure of the *epistola posterior*.

Immediately after the first anagram, Newton proceeded to explain why the method for drawing tangents was important to him:

> On this foundation I have also tried to simplify the theories which concern the squaring of curves, and I have arrived at certain general Theorems.[67]

This is the crucial point for Newton. Thanks to the understanding of the inverse relation between the method for drawing tangents (differentiation) and that for squaring curves (integration), a general method for drawing tangents translates into a powerful tool for constructing (in Leibnizian terms) integral tables. Indeed, this is the method (Method 2) that allowed the compilation of the first catalogue of curves of *De Methodis*. Newton did not want to reveal this method. He was not much concerned with his solution of Problem 1 and its application to the direct method of fluxions: determination of tangents, curvatures, maxima and minima, and the like. He was rather concerned with the results on the inverse method achieved in tackling Problem 2. And, indeed, just after the first anagram he proceeded to show the prime theorem on quadratures (§8.5.1) and its applications to three examples. The prime theorem is a generalization of results contained in the first catalogue of curves of *De Methodis*.[68] Newton communicated to Leibniz some of his theorems on quadratures achieved via Method 2 but not the actual method that he had employed.[69]

Following that, Newton hinted at theorems for the quadrature of curves by "comparison with conic sections" that he had achieved "withdrawing from the contemplation of figures," that is, "reducing the whole matter to the simple consideration of ordinates alone." This is Method 3, which allowed the compilation of the second catalogue of curves, whereby by a transformation of variables (the "contemplation of ordinates") the areas of surfaces subtended to some difficult curves are calculated in terms of conic areas. On this method Newton kept everything hidden. He merely listed a few curves taken from the second catalogue that, he claimed, could be squared by Method 3.[70]

---

involving the fluxions of quantities is exhibited, to determine the relation of the quantities one to another." MP, 3, p. 83. See §8.3.1 and §8.4.1.

67 *Correspondence*, 2, p. 134.

68 See the discussion in MP, 3, p. 237, n. 540.

69 He wrote, "And to be frank, here is the first Theorem." *Correspondence*, 2, p. 134. But how could even a mathematician of Leibniz's stature guess the demonstration?

70 *Correspondence*, 2, p. 117, pp. 119–20.

The second anagram, which occurs at the end of the *epistola posterior*, also has to do with the inverse method, namely, with the inverse method of tangents (in modern terms, integration of ordinary differential equations).[71] Once again, Newton hid his secrets behind an anagram. It is interesting to consider how he introduced the second anagram because his wording reveals his priorities and his policy of publication:

> When I said that almost all problems are soluble I wished to be understood to refer specially to those about which mathematicians have hitherto concerned themselves, or at least those in which mathematical arguments can gain some place. For of course one may imagine others so involved in complicated conditions that we do not succeed in understanding them well enough, and much less in bearing the burden of such long calculations as they require. Nevertheless—lest I seem to have said too much—inverse problems of tangents are within our power, and others more difficult than those, and to solve them I have used a twofold method of which one part is neater, the other more general. At present I have thought fit to register them both by transposed letters, lest, through others obtaining the same result, I should be compelled to change the plan in some respects.[72]

The meaning of the second anagram is as follows:

> One method consists in extracting a fluent quantity from an equation at the same time involving its fluxion; but another by assuming a series for any unknown quantity whatever, from which the rest could conveniently be derived, and in collecting homologous terms of the resulting equation in order to elicit the terms of the assumed series.[73]

Leibniz would not have been helped by decoding the second anagram either. Newton introduced the second anagram by making clear that it hid a double method that allowed the solution of the most cutting-edge problems considered by the mathematicians of his times. Further, he openly declared himself to be particularly proud of this advanced double method.

One can gather that these two methods for the resolution of the inverse method of tangents correspond to Method 1 (§8.4.2) of *De Methodis* and to the method of undetermined coefficients that was to play such an important role in *De Quadratura* (Theorem 3; §8.5.2).[74] The method of undetermined coefficients underpins both

[71] Recall that in the inverse problem of tangents, the property of the tangent of a curve is given, and one has to determine the curve. Of course, this corresponds to integrating a differential equation.
[72] *Correspondence*, 2, p. 148.
[73] *Correspondence*, 2, p. 159.
[74] This interpretation is strongly supported (§17.3) by the comment to the second anagram in Newton's "Account" (1715), p. 193. In his commentary to the second anagram, Wallis affirmed that the interpretation of the second method was easy: "Harum methodorum Secunda ex verbis jam recitatis absque ulteriore explicatione intelligi potest." *Opera*, 2, p. 393.

the theorems of *De Quadratura* and the approximations of curvilinear areas via interpolation of *Methodus Differentialis* (§8.6).

From this analysis of the *epistolae* it emerges that Newton was open insofar as the quadrature techniques of *De Analysi* are concerned. The fountain of knowledge that in the mid-1670s he kept away from Collins and Leibniz consisted in the more advanced quadrature techniques of *De Methodis* and those that he was developing in writings that would eventually lead to *De Quadratura*.

16.3.7 Leibniz's Second Reply, June 11/21, 1677

But Leibniz knew about this fountain very well as he showed in his reply dated June 11/21, 1677. Newton must have been chilled by what he found there.[75] The bold and reassuring words he had sent to Collins a few months before ("as for ye apprehension yt M. Leibnitz's method may be more general or more easy then mine, you will not find any such thing") were disproved.[76] Leibniz presented the direct algorithm in full detail, going beyond Sluse's method exactly as Newton had done, his differential calculus being applicable not only to polynomials but also to quantities involving quotients and roots. Leibniz was right on target when, after presenting the rules and the application of the differential method, he added, "In my opinion, what Newton wished to conceal about the drawing of tangents is not discordant with these [rules]."[77]

Further, and this must have been more worrying for Newton, Leibniz immediately seized the gist of Newton's first anagram by showing full understanding of the fundamental theorem of the calculus:

> What he [Newton] adds [just after the first anagram], on this same basis, that squarings are also rendered easier, confirms me in this opinion: those figures surely are always quadrable [read "integrable in finite terms"] which are related to a differential equation.[78]

Leibniz continued showing applications of integration via antidifferentiation, exactly as Newton had done with Method 2 in the first catalogue of curves.

Leibniz wrote another letter to Oldenburg on July 12/22, 1677, in continuation of the letter dated June 11/21.[79] At this crucial point Newton interrupted the correspondence. Leibniz was getting too close to the secret fountain of his advanced methods for squaring curves. The correspondence between Newton and Leibniz lagged until March 7/17, 1692/93, when Leibniz approached Newton with much

[75] *Correspondence*, 2, pp. 212–9.
[76] Newton to Collins (November 8, 1676). *Correspondence*, 2, p. 179.
[77] *Correspondence*, 2, p. 221.
[78] *Correspondence*, 2, p. 221.
[79] *Correspondence*, 2, pp. 231–2.

politeness. Newton replied on October 16, 1693, apologizing for his protracted silence. He also expressed his worries that Leibniz might be offended by the imminent publication of the second volume of Wallis's *Opera*, where the fluxional method and notation were briefly presented, and he deciphered the first anagram. Newton thus abandoned the strategy followed in the mid-1670s; he was now determined to let something of his fluxional quadratures be printed.[80]

16.4 Disclosures, 1684–1699

From the mid-1680s, Newton realized that the policy of scribal publication that he had adopted in the 1670s had to be revised. In introducing the second anagram in the *epistola posterior* he had stated that he would have revealed the higher methods of quadrature if others were "obtaining the same result."[81] This is exactly what happened in the 1680s and 1690s. Developments in quadrature techniques obtained, for example, by David Gregory, John Craig, and, on the Continent, by Leibniz posed a direct challenge to Newton. Therefore he began disclosing knowledge about his analytical method of fluxions, even about his higher quadrature methods.

In 1684, David Gregory entered Newton's life in a traumatic way. It was traumatic precisely because in Gregory's letter dated June 9 he announced the *Exercitatio Geometrica de Dimensione Figurarum*, a work where David Gregory edited his uncle James Gregory's results on squaring via infinite series.[82] After receiving the *Exercitatio Geometrica*, Newton began writing a dossier on quadratures, "Matheseos Universalis Specimina," which incorporated material from his two *epistolae* for Leibniz.[83] Most probably, it is because of Halley's eventful visit, which sparked interest in planetary motions and the writing of the *Principia*, that Newton abandoned the risposte to Gregory. It would be simplistic to think that Newton meant "Matheseos Universalis Specimina" for print publication, as is often surmised. Given its forensic character (it consists of a collection of excerpts from letters interspersed with information on the circumstances that led to their composition and annotations on their mathematical content), it is more probable that Newton conceived it as a dossier to be circulated or shown to visitors.

In 1685, John Craig visited Newton (§8.5.1, §16.2). It is interesting to note that at this juncture, while abandoning the project of completing "Matheseos Universalis Specimina," Newton revealed to Craig one of the pillars of his higher quadrature

[80] *Correspondence*, 3, pp. 257–8, pp. 285–6.
[81] "I have thought fit to register them [Newton's two methods of quadratures] both by transposed letters, lest, through others obtaining the same result, I should be compelled to change the plan in some respects." *Correspondence*, 2, p. 148.
[82] Gregory to Newton (June 9, 1684) letter accompanying *Exercitatio Geometrica de Dimensione Figurarum* (1684) on series expansion of $(1 \pm x)^{\pm 1/2}$. *Correspondence*, 2, p. 396.
[83] Add. 3964.3, ff. 7r–20v, edited by Whiteside in MP, 4, pp. 526–89.

methods, the prime theorem. Craig was close to publishing his pioneering treatise on quadratures, *Methodus Figurarum Lineis Rectis & Curvis Comprehensarum Quadraturas Determinandi* (1685). This short treatise was written in differential notation and gave credit to a number of mathematicians. Newton was cited in passing, together with Descartes, Fermat, Sluse, Barrow, Wallis, Ehrenfried Walther von Tschirnhaus, and Leibniz. The same holds true for Craig's next work, *Tractatus Mathematicus de Figurarum Curvilinearum Quadraturis et Locis Geometricis* (1693). Newton's decision to reveal the prime theorem to Craig finds its justification in his awareness that Gregory, Craig, and Leibniz were progressing toward the discovery of his higher quadrature methods. The prime theorem fell into the hands of David Gregory, who printed it as one of his own discoveries in Pitcairn's *Solutio Problematis de Historicis seu Inventoribus* (1688) (§8.5.1).

In writing the *Principia*, Newton continued to reject disclosure in print of his analytical method of quadratures. Part IV considered Newton's publication policy of his mathematics in the *Principia*, a work in which he had to assemble a panoply of mathematical tools, from quadratures to infinite series, from porismatic geometry to first and ultimate ratios, behind a seemingly coherent façade. As mentioned, the more competent readers understood the might of Newton's mathematical achievement but were frustrated by the omissions that they identified in the text. The mathematical subtext that Newton kept away from his readers was what seemed interesting to experts such as Huygens and Johann Bernoulli. What were the analytical tools that Newton possessed and that lay behind his geometrical constructions? This question recurred often in the correspondence of the mathematicians who set themselves the challenging task of understanding Newton's *magnum opus*.[84] Recall that its readers had access to the binomial theorem (stated in the Scholium to Proposition 93, Book 1) and to infinite power series expansions (Proposition 45, Book 1, and Proposition 10, Book 2). Newton was eager to communicate his results on infinite series. Algebraic and fluxional equations, which were deployed in certain demonstrations, were neglected (see chapters 11 and 12). Some knowledge of the foundations of the method of fluxions could be derived from the synthetic versions provided in Section 1, Book 1, and in Lemma 2, Book 2 (§9.3, §9.4). But no indication on how the algorithm might work was given. Such information was provided only partially and posthumously in an appendix, entitled "Explications, (given by a Friend,) of some Propositions in this Book, not demonstrated by the Author," to the English translation of the *Principia* by Andrew Motte (1729).

In 1691, Gregory wrote to Newton claiming the prime theorem on quadratures (§8.5.1), which he had learned from Craig's recollection of the visit to Newton

[84] Guicciardini, *Reading the Principia* (1999).

in 1685, as his own invention.[85] Gregory was challenging Newton on the inverse method. Newton's reaction was one of panic and rage. He immediately set himself the task of writing extensively on quadratures, and he did so with characteristic historicism, his aim being that of showing his priority in these important matters.[86] The first drafts of what became *De Quadratura* were written in this context. The direct method of tangents was not as important as the inverse method of quadratures, from Newton's point of view; his reaction to Sluse's paper on the direct method was lukewarm (§16.3.2).

In May 1694, despite these tensions, and for reasons unclear to me, David Gregory was admitted into Newton's close circle. Rather than printing his fresh treatise on quadratures (a shortened version appeared as an appendix to the *Opticks* in 1704), the Lucasian Professor decided to show his private manuscripts to Gregory in his rooms at Trinity College. After these eventful *privatissima* Gregory became a faithful acolyte. He recorded his encounters with Newton in extensive memoranda and wrote a short tract on fluxions (§16.2). He also speculated on the idea of transforming his "Notae" on the *Principia* into a running commentary—modeled on van Schooten's exemplary commentary to Descartes' *Géométrie*—for a second edition of the *magnum opus*. Meeting Newton's favor could be profitable in establishing a reputation in the diplomatic competition that was inflaming the European Republic of Letters.[87]

Typically, Gregory's memoranda said very little about the rules of the direct method, whereas he praised Newton's inverse method for the quadrature of curves. For instance, in 1694, Gregory stated, "The problem of quadratures and the inverse method of tangents includes the whole of more advanced geometry."[88] Gregory was impressed by Newton's catalogues of curves and recognized that these quadratures underpinned many of the most advanced parts of the *Principia*:

> The second treatise [*De Quadratura*] will contain his [Newton's] Method of Quadratures ... on these [quadratures] depend certain more abstruse parts in his philosophy as hitherto published, such as Corollary 3, Proposition 41 and Corollary 2, Proposition 91.[89]

If one turns to Nicolas Fatio de Duillier, the most influential of Newton's acolytes in the 1690s, one discovers that when he acted as a middleman between Huygens

---

[85] Gregory to Newton (November 7, 1691). *Correspondence*, 3, pp. 172–6. This letter was sent to Wallis for inclusion in his *Opera*, 2 (1693), pp. 377–80.
[86] See Newton's draft reply to David Gregory in MP, 7, pp. 21–3.
[87] See Guicciardini, *Reading the Principia* (1999), pp. 179–84. The similarity with van Schooten's commentary is particularly evident in the exemplar held at Christ Church Library.
[88] *Correspondence*, 3, p. 313. "Problema de Quadraturis et Methodo tangentium inversa omnen reconditiorem Geometriam comprehendit." David Gregory's memorandum of a May 1694 visit to Newton. Translation in *Correspondence*, 3, p. 318.
[89] Edinburgh University Library, MS Gregory C42. David Gregory's memorandum (July 1694) of a May 1694 visit to Newton. Translation in *Correspondence*, 3, p. 386.

and Newton in the years immediately following the publication of the *Principia*, he had an extensive correspondence with his former Dutch mentor about the inverse method of tangents. This is what Huygens wished to know. In the correspondence between Fatio and Huygens little attention is paid to the direct method.[90] Fatio was clearly capitalizing on his privileged access to Newton's conversation and manuscripts; indeed, his status among the European mathematicians was high at that juncture exactly because he could inform correspondents about the subtexts hidden in Newton's printed work.

It is thanks to Wallis that Newton's avoidance of printed mathematical publication began to fade. Newton allowed Wallis to print excerpts from his fluxional manuscripts and letters. This was, at this juncture, a reasonable compromise. Newton used Wallis's *Algebra* and *Opera* as vehicles for rendering his analytical method of fluxions better known while avoiding an involvement in an authored publication, such as a signed paper or book. Wallis, always keen on making available in print analytical heuristic tools, especially those achieved in Britain, was able—after imploring Newton in several letters—to obtain accounts of the new analytical method, which he inserted in his English *Algebra* (1685) and in his Latin *Opera* (vol. 2, 1693, vol. 1, 1695, vol. 3, 1699). Newton allowed Wallis to print the *epistola prior* almost in its entirety and material taken from the *epistola posterior* in the English *Algebra*; the binomial theorem and some of the quadratures via infinite series of *De Analysi* were printed there.[91] For the second volume of the *Opera*, Wallis obtained the *epistola posterior* and additional material provided by Newton; the first presentation in print of the method of fluxions appeared on pages 390–396. Wallis provided a synopsis of the *epistola posterior* and included the deciphering of the string anagrams (on pp. 392–393) and an explanation, missing from the *epistola posterior*, of the dotted notation for fluxions. The full text of the two *epistolae* was eventually printed in the third volume of the *Opera* (1699). These events were to play a momentous role in the controversy with Leibniz. What appeared in Wallis's *Opera* was, however, a small fraction of Newton's mathematical output and certainly did not exhaust the appetite of those who were trying to divine the subtext of the *Principia*.

The challenge originated by the development of calculus in Britain and on the Continent, combined with the effects of the uncontrolled dissemination of Newton's mathematical manuscripts, induced Newton to print his analytical method of fluxions (see chapter 17).

---

[90] On the correspondence between Fatio and Huygens on the inverse method, see Vermij and van Maanen, "An Unpublished Autograph by Christiaan Huygens" (1992).

[91] Excerpts from the 1676 *epistolae* to Leibniz in *Algebra*, pp. 330-46; the content of the *epistolae* is reviewed in *Opera*, 2, pp. 368–396, where Wallis could rely upon material delivered from Newton (the prime theorem on quadratures as part of a draft of *De Quadratura* is on pp. 390–6); full text of the 1676 *epistolae* to Leibniz in 3, pp. 622–9, 634–45.

17 Fluxions in Print, 1700–1715

> In a Letter written to Mr. Leibnitz in the Year 1676, and published by Dr. Wallis, I mentioned a Method by which I had found some general Theorems about squaring Curvilinear Figures, or comparing them with the Conic Sections, or other simplest Figures with which they may be compared. And some Years ago I lent out a Manuscript containing such Theorems, and having since met with some Things copied out of it, I have on this Occasion made it publick . . . . And I have joined with it another small Tract concerning the Curvilinear Figures of the Second Kind, which was also written many Years ago, and made known to some Friends, who have solicited the making it publick.
>
> —Isaac Newton, 1704

17.1 In the Public Sphere

17.1.1 The Demise of Newton's Scribal Strategies, 1700–1703

Newton's policy of controlled scribal publication was no longer tenable at the turn of the seventeenth century. In 1699, Fatio had bluntly posed the question of priority in his *Lineae Brevissimi Descensus Investigatio Geometrica*, noting that on the Continent the calculus was unjustly attributed to Leibniz. As a privileged acolyte of Newton he informed his readers that his great mentor had devised an equivalent algorithm well before the publication of Leibniz's "Nova Methodus." In 1695, Wallis had complained to Newton,

> [Y]our Notions (of *Fluxions*) pass there with great applause, by the name of *Leibniz's Calculus Differentialis* . . . You are not so kind to your Reputation (& that of the Nation) as you might be, when you let things of worth ly by you so long, till others carry away the Reputation that is due to you.[1]

The challenge for Newton came not only from the Continent but also from Britain (§16.4). Manuscripts were circulating in a rather uncontrolled way. As Newton affirmed in the advertisement to the *Opticks* (1704), he was going to add *De Quadratura* and the *Enumeratio* as appendices, since there were too many "things copied out" of a manuscript that had been lent out and "made known to some Friends."[2]

Epigraph from Newton, *Opticks* (1704), Advertisement [n.p.].

[1] Wallis to Newton (April 10, 1695) in *Correspondence*, 4, p. 100. See also similar complaints by Wallis in his letter to Newton (April 30, 1695) in *Correspondence*, 4, pp. 116–7.

[2] Newton, *Opticks* (1704), Advertisement [n.p.].

Further, British mathematicians such as Gregory and Craig were producing results that were as worrying for Newton as those of Leibniz and the Bernoullis.

The situation came to a head because of a tract authored by an amateurish mathematician, the iatro-mechanist George Cheyne, who tried with his *Fluxionum Methodus Inversa* (1703) to write a treatise completely devoted to the inverse method. He was induced to attempt this task by his influential friend Pitcairn, who had already caused frictions with Newton by hosting David Gregory's quadratures in his *Solutio Problematis de Historicis seu Inventoribus* (1688). The result was, as Whiteside puts it, "a competent and comprehensive survey of recent developments in the field of inverse fluxions not merely in Britain, at the hands of Newton, David Gregory, and John Craig, but also by Leibniz and Johann Bernoulli on the Continent, and drew the assemblage together and systematized it with proofs and elaborations of Cheyne's own contrivance."[3] Cheyne's pamphlet is representative of the dependence of British authors on continental work on integration that was prevalent in this period.[4] Craig and Gregory found inspiration on quadrature techniques in the papers by Leibniz and the Bernoullis printed in *Acta Eruditorum*. Newton was obviously displeased by Cheyne's *Fluxionum Methodus Inversa*. Abraham De Moivre was instructed to attack Cheyne, and he did so with vehemence in his *Animadversiones in D.Georgii Cheynaei Tractatum de Fluxionum Methodo Inversa* (1704).

The fact that there was some danger in proposing a book on Newtonian discoveries without previous negotiations with Newton himself is evident not only from the Cheyne affair but also from Humphry Ditton's *The General Laws of Nature and Motion* (1705), a small treatise in which the English reader was given some instructions on the first three sections of the *Principia*. In the *Preface*, Ditton was cautious in stating Newton's absolute right of property on the mathematization of central forces:

> The materials that this Book is composed of, are so absolutely Mr Newton's Property, that I dare hardly pretend to call any thing mine. The Principles most certainly are all his own: and if I have attempted any where to make any Use of them, or to draw any consequences from them; yet the indisputable Right that he has to the Former, gives him a title to the Latter also, where they are just and good. This is certain, that his Inventions are new and compleat; and equally exclude all the Additions and Claims of those that come after. ... Further, To render what I have done more universally serviceable here at Home, I chose to make it appear in English rather than Latin. For if it be granted that Mr. Newton's Discoveries are but barely useful, there's no Reason why a Multitude of very capable Minds

[3] MP, 8, pp. 17–8

[4] On the influence of the *Acta Eruditorum* on David Gregory, see Eagles, *The Mathematical Work of David Gregory* (1977). An interesting paper covering the Continental influence on British mathematicians is Schneider, "Direct and Indirect Influences" (2006).

> shou'd be debarr'd from them meerly [sic] for want of a Language. ... Thus indeed I confess, do some people argue for keeping the Sacred Books in an unknown Tongue: But we pretend to a Protestant Liberty, at least with respect to our Philosophy.[5]

In the meantime Newton became aware that it was time to publish his treatise on quadratures written in 1693. He appended it (enriched with a newly written Introduction and Scholium) to the first edition of the *Opticks* (1704) with the title *Tractatus de Quadratura Curvarum*. In the appendix to the *Opticks* one could also find *Enumeratio Linearum Tertii Ordinis*. The printing stage for Newton as a mathematical author begins properly in 1704. It is interesting to consider which works, among the vast number of the mathematical manuscripts that he kept in his hands and circulated among his acolytes, Newton chose to print; the circumstances in which his decisions were taken; and the editorial choices he made before printing them. My aim in this chapter is to examine the priorities and values that guided him when he moved from scribal to printed publication.

17.1.2 The Two Treatises of 1704

The great event in Newton's career as a mathematical author occurred in 1704 when the *Opticks* was printed. At last the reader could hope for more light on his methods. Two short mathematical tracts were appended, the *Enumeratio* and *De Quadratura*. The high hopes were, however, soon frustrated.

As discussed in chapter 6, the *Enumeratio* lacked all the proofs concerning the reduction of third-degree equations to four basic forms, the methods for plotting the cubics complex graphs that enigmatically adorn the beautifully engraved plates, the projective classification of cubics into five classes, and the organic generation through given points. The interested reader could only surmise that the *Enumeratio*, like the *Principia*, had its well-hidden subtext.

De Quadratura was an equally challenging text. Newton's concise presentation of his complicated quadratures needed commentary and explanation. Commentaries to *De Quadratura* therefore flourished in the first half of the eighteenth century. Explaining *De Quadratura* was a frequently attempted task for Newton's mathematical followers. Careful recalculations of Newton's theorems in William Jones's hand, probably intended for his patron or for the use of his students, have survived.[6] In 1745, John Stewart still thought it worthwhile to explain Newton's theorems in more than 400 pages of commentary, and some 20 years later Le Seur and Jacquier devoted two chapters of their *Elemens du Calcul Intégral* (1768) to a careful analysis of *De Quadratura*. It was agreed that such an effort was necessary because

[5] Ditton, *The General Laws of Nature and Motion* (1705), *Preface* [n.p.].
[6] Cambridge University Library, Macclesfield Collection, Add. 9597.4.37.

in order to understand the *Principia* fully one had to understand the complicated techniques of *De Quadratura*.[7] Stewart wrote,

> But further, every one, who is the least conversant in these Matters, knows, that natural Philosophy never was, nor can be successfully prosecuted or advanced, but by the Help of Geometry and Arithmetick ... Witness our noble Author, who in that admirable Performance, his Mathematical Principles of Natural Philosophy, hath happily shown us, of what great Use abstract Mathematical Knowledge may be, for investigating the Forces of natural Bodies ... and as the great part of the Discoveries contained in that Book, is owing to and founded upon, the Doctrine of Fluxions ... as was observed some time ago by the noble and learned Marquis de l' Hospital; so we find in particular, that he often proceeds upon the Quadrature of Curves as a Postulatum, or Principle already known and granted. See Propos. 46, 53, 54, 56, 81, Book I and many other Places. By which he has shewn, that the most sublime parts of geometry, and particularly the Doctrine of Fluxions, and the Quadrature of Curves, are of infinite Use in true Philosophy.[8]

In 1712 and 1719, Newton thought about adding *De Quadratura* as an appendix to the *Principia*.[9]

Both the *Enumeratio* and *De Quadratura* presented challenging, somewhat enigmatic, and cutting-edge research. They appealed to Newton because by printing them he could display the level of his results in areas that he very much cherished: projective geometry and organic constructions (in the *Enumeratio*) and the inverse method of fluxions (in *De Quadratura*). These were the results that Newton was at this point anxious to establish as his own achievement, in a context in which the Leibnizian calculus was becoming prominent on the Continent and Britain. Note that Newton did not print what appears to us to be his masterpiece, *De Methodis*, where the direct method of fluxions was explained and applied to tangents and curvatures. Most probably he was not interested in making public a treatise written "for the use of learners," considering it too elementary to deserve print publication.[10]

---

[7] The two Minim Friars Thomas Le Seur and François Jacquier, who had promoted knowledge of the *Principia* with a detailed commentary (1739–42), devoted their *Elemens du Calcul Intégral* (1768), Chapters 4–7 of the first volume (pp. 135–544), and Chapter 2 of the second volume (pp. 52–103), to a commentary on Newton's methods of quadrature in *De Quadratura* and *De Methodis*.

[8] Newton, *Two Treatises* (1745), p. viii. The reference to L'Hospital, *Analyse des Infiniment Petits* (1696), is in fact to the Preface, which is due to Fontenelle.

[9] Copies of *De Quadratura* recast by Newton for publication as an appendix to the *Principia* are extant. MP, 8, pp. 258ff, pp. 625ff, pp. 656ff. In a letter from Cotes to Newton written on April, 26, 1712, one reads: "I am glad to understand by Dr. Bentley that You have some thoughts of adding to this Book [the *Principia*] a small Treatise of Infinite Series & the Method of Fluxions." *Correspondence*, 5, p. 279; Cohen, *Introduction to Newton's Principia* (1971), pp. 238–9.

[10] As John Colson made clear in his Preface to the English translation of *De Methodis*, which appeared in 1736: "[The *Method*] is of an elementary nature, preparatory and introductory to

17.1.3 *Arithmetica Universalis*, 1707

The circumstances surrounding the publication of *Arithmetica Universalis* (1707) are interesting (§4.1). In this work, devoted to Cartesian algebra, Newton published results on the theory of equations that he had achieved in the 1670s and deposited at the University Library in 1684. *Arithmetica Universalis*, which had a didactic structure and which rendered the algebraic analysis wholly explicit, appeared anonymously in 1707. Newton made it clear that he was compelled to publish in order to obtain the support of his Cambridge colleagues in the election to the 1705 Parliament.[11] In the opening "To the Reader" of the first English translation it was stated that the author had "condescended to handle" the subject.[12] *Arithmetica Universalis* also ended with statements in favor of geometrical method and against the moderns who had lost the elegance of geometry (§4.2). The fact that its author did not recognize *Arithmetica Universalis* as representing what he meant by good geometry, which instead was the style of the *Principia*, was made clear to any person belonging to the Republic of Letters.

17.1.4 Jones's Edition of Newton's Mathematical Tracts, 1711

A second wave of mathematical printed publication of Newton's works occurred in the heated context of the dispute with Leibniz. In 1711, William Jones, who in 1712 was to play a decisive role as a member of the committee of the Royal Society since he was in possession of Collins's papers, edited *Analysis per Quantitatum Series, Fluxiones, ac Differentias: cum Enumeratione Linearum Tertii Ordinis*. In this lavishly produced booklet one can find *De Analysi*, *De Quadratura*, the *Enumeratio*, and *Methodus Differentialis*, together with excerpts from some of Newton's letters, later reproduced in full with much additional material in *Commercium Epistolicum*.[13] Three features of Jones's edition should be briefly discussed.

---

his other most arduous and sublime Speculations, and intended by himself for the instruction of Novices and Learners." Colson also stated that "Pemberton, as he acquaints us in his *View of Sir Isaac Newton's Philosophy*, had once a design of publishing this Work, with the consent and under the inspection of the Author himself." Newton, *The Method of Fluxions* (1736), pp. ix–x.

[11] David Gregory wrote, "He was forced seemingly to allow it, about 14 months agoe, when he stood for Parliament-man at the University. He has not seen a sheet of it, nor knows he what value it is in, nor how many sheets it will make, nor does he well remember the contents of it. He intends to goe down to Cambridge this summer and see it, and if does not please him to buy up the copyes." Hiscock, *David Gregory, Isaac Newton and Their Circle* (1937), p. 36.

[12] "If any Thing could add to the Esteem every Body has for the Analytick Art, it must be, that Sir Isaac has condescended to handle it." Newton, *Universal Arithmetick* (1720), p. i.

[13] Namely, fragments of the two 1676 *epistolae* for Leibniz, the material Newton sent to Wallis to be inserted in the second volume of the *Opera* (1693), and the quadrature-letter to Collins (1676). Newton, *Analysis per Quantitatum* (1711), pp. 23–38.

Figure 17.1
Engraving from Newton, *Analysis per Quantitatum, Series, Fluxiones, ac Differentias: cum Enumeratione Linearum Tertii Ordinis* (1711). This engraving is placed before the title of *De Analysi*. It expresses a cherished idea of Newton and his acolytes. The fluxional methods revealed in *De Analysi* would, according to Jones's propaganda, constitute the hidden analysis of the *Principia*. Indeed, the mythological characters display scrolls and shields where one can distinguish some of the diagrams of the *Principia*'s key propositions. One recognizes, from left to right, several diagrams that occur in Book 1: (on the ground) Prop. 94 on the motion of light corpuscles refracted by a medium; (held by a putto) Prop. 66 on the three body problem; (on the shield) Prop. 32 on rectilinear fall accelerated by an inverse-square force and Prop. 43 on precession of orbits; (on the ground) Prop. 1 on the law of areas; (on the ground) Cor. 2, Prop. 91, on the attraction exerted by an oblate ellipsoid. The message addressed to the Leibnizians such as Johann Bernoulli could not be clearer. Source: Newton, *Analysis per Quantitatum* (1711), p. 1. Courtesy of the Biblioteca Universitaria di Bologna.

1. The stress is on quadrature techniques. *De Analysi* is defined as a "short treatise on the quadrature of curves," its use being to allow quadratures via series expansion.[14] The excerpts from the correspondence are equally concerned with the inverse method. Nowhere does Jones insist on the direct method as an interesting topic.
2. Newton's early use of infinitesimals is obliterated by a skillful editing of *De Analysi*. Most notably, William Jones changed, probably in accordance with Newton's instruction, occurrences of *esse infinite parvam* into *in infinitum diminui & evanescere*.[15] These interventions were aimed at changing the orig-

[14] "brevis de Curvarum Quadratura Tractatus." Newton, *Analysis per Quantitatum* (1711), Praefatio [n.p.].
[15] Newton, *Analysis per Quantitatum* (1711), p. 20.

Figure 17.2

Engraving from Newton, *Analysis per Quantitatum, Series, Fluxiones, ac Differentias: cum Enumeratione Linearum Tertii Ordinis* (1711). This engraving is placed before the title of *Enumeratio Linearum Tertti Ordinis*. It expresses an idea that was quite important for Newton. The objects of geometry, in this case, plane curves, should be conceived of as traced by motion, geometry being based upon mechanical practice. Indeed, the two putti are intent on drawing a conic by deploying Newton's mechanical description via rotating rulers (§5.4). Such generations of curves, treated in the final pages of the *Enumeratio*, can be performed both by artifice and by nature (see chapters 13 and 14). Plane curves are indeed daily observed in *rerum natura*. On the right, Urania (?) has at her feet a representation of the circular orbit of the moon, a curve that can also be traced by means of the compass held by a putto. With her right hand she holds a representation of the elliptical orbit traced by a comet, which one of the putti is generating "organically" by means of the instrument devised by Newton. Geometry, when conceived of as devoted to kinematically generated objects, paves the way for astronomy (represented by two books and the armillary sphere toward which Urania points her finger). Source: Newton, *Analysis per Quantitatum* (1711), p. 69. Courtesy of the Biblioteca Universitaria di Bologna.

inal form of the method of fluxions into a version more homogeneous with the *Principia* and *De Quadratura*, which were based on limits.[16]

3. The engravings that adorn Jones's edition of Newton's mathematical tracts visually express another cherished idea of Newton and his acolytes. According to Jones's propaganda, the analysis revealed in these tracts constitutes the hidden

[16] As Jones stated, "Hujus Geometriae Newtonianae non minimam esse laudem duco, quod dum per limites Rationum Primarum & Ultimarum argumentatur, aeque demonstrationibus Apodicticis ac illa Veterum munitur; utpote quae haud innititur duriusculae illi Hypotesi quantitatum Infinite parvarum vel Indivisibilium, quarum Evanescentia obstat quominus eas tanquam quantitates speculemur." Newton, *Analysis per Quantitatum* (1711), Praefatio [n.p.] This, however, strictly applies to *De Quadratura*, not to the original *De Analysi*.

Figure 17.3
Engraving from Newton, *Analysis per Quantitatum, Series, Fluxiones, ac Differentias: cum Enumeratione Linearum Tertii Ordinis* (1711). This engraving is placed before the title of *Introductio ad Quadratura Curvarum*. The quadrature techniques of the analytical inverse method of fluxions are illustrated by a workshop where mathematics directs the diverse and purposeful activities of craftsmen. The putti make use of instruments such as hammers, saws, chisels, brushes, furnaces, and balances. Useful applications are evoked, such as carpentry, painting, fortification (alluded to in the drawing at the upper right corner). Here we see the method of quadrature represented as a collection of disconnected, heuristic techniques, as a craft rather than as a unified theory. A tension surfaces between this image and previous images, which stress continuity between the mathematical tracts edited by Jones and astronomy. In the years of the polemic with Leibniz, Newton often stressed the idea that the analytical method of series and fluxions is a heuristic technique that should not be confounded with the "good geometry" on which the study of the "system of the heavens" is grounded. On the other hand, he maintained that the propositions of the *Principia* had been found by application of the "new analysis." Source: Newton, *Analysis per Quantitatum* (1711), p. 41. Courtesy of the Biblioteca Universitaria di Bologna.

analysis of the *Principia*. Indeed, the mythological characters represented in one of the engravings display scrolls and shields where one can distinguish some of the diagrams of the *Principia*'s key propositions (figure 17.1). Two other engravings can be interpreted as illustrations of other important aspects of Newton's conceptions of mathematics, namely, his ideas on the relations between mechanics, geometry, and natural philosophy (figure 17.2), and his evaluation of the analytical method of fluxions as a heuristic technique (figure 17.3).

17.2 *Commercium Epistolicum*

Jones's edition of Newton's mathematical tracts anticipated many aspects of *Commercium Epistolicum* (1713). Also in *Commercium Epistolicum*, *De Analysi* was presented as the main proof that Newton was the first inventor of the method of fluxions, and ample use was made of Newton's correspondence with Collins and Oldenburg held in the archives of the Earl of Macclesfield and at the Royal Society.

Commercium Epistolicum can be considered Newton's last mathematical work. I have always been fascinated by the technical, both mathematically and forensically, terminology that Newton employed in this pamphlet and by its carefully drafted structure. There is no doubt, in my opinion, that Newton believed himself to be right in his quarrel with Leibniz and that he was convinced of having provided indisputable evidence of the German's plagiarism.

But there is a puzzle here. There is a general consensus among historians siding with Leibniz, such as Jean Etienne Montucla, Jean-Baptiste Biot and Felix Lefort, and Augustus De Morgan, that *Commercium Epistolicum* failed to accomplish what Newton meant to achieve, that is, to deliver convincing proof of Newton's case against Leibniz.[17] The failure of *Commercium Epistolicum* is supposedly total; it is claimed that it does not even yield evidence that Newton discovered the calculus before Leibniz, since it deals with topics deemed to be only loosely related to the calculus. Most notably, the two *epistolae* that Newton addressed to Leibniz in 1676, and that were meant to constitute the main proof provided in *Commercium Epistolicum* that crucial information was passed to the German, were often described as lacking this very evidence. Leibniz's remonstration, expressed in the posthumous "Historia et Origo Calculi Differentialis," has been considered justified and correct by many commentators,

> They have changed the whole point of the controversy, for in their publication ... one finds hardly anything about the differential calculus; instead every other page is made up of what they call infinite series ... This is certainly a useful discovery, for by it arithmetical approximations are extended to the analytical calculus; but it has nothing at all to do with the differential calculus. They use this sophism, that whenever his adversary works out a quadrature by addition of the parts by which a figure is gradually increased, at once they hail it by the use of the differential calculus [as, for instance, on page 15 of *Commercium Epistolicum*; see figure 7.5] ... Since therefore his opponents, neither from the *Commercium Epistolicum* that they have published, nor from any other source brought forward the slightest bit of evidence whereby it might be established that his rival used this calculus before it was published by our friend; therefore all the things that they have reported may be rejected as extraneous to the matter. They have made recourse to the skill of ranters with the purpose of diverting attention of judges from the matter on trial to other things, namely to infinite series.[18]

[17] See the apparatus criticus by Biot and Lefort in their 1856 edition of *Commercium Epistolicum*. I cite from Montucla's *Histoire* (1799–1802). De Morgan authored several revisionist papers concerning the priority dispute. See De Morgan, "On the Additions Made to the Second Edition of the *Commercium Epistolicum*" (1848), "A Short Account of Some Recent Discoveries" (1852), and "On the Authorship of the Account" (1852).

[18] "Mutarunt etiam statum controversiae, nam in eorum scripto ... de calculo differentiali vix quicquam [invenitur]: utramque paginam faciunt series, quas vocant, infinitae. ... Utile est inven-

The defensive policy followed in *Commercium Epistolicum* might appear, and to Leibniz surely did appear, as a clever way of shifting the level of discourse in order to avoid a fair confrontation. But after the publication of Newton's mathematical papers it is clear that Newton might have provided very many sources as evidence of his use of an algorithm equivalent to his rival's differential calculus. Why did Newton fail to do so? Why did he engineer *Commercium Epistolicum* to focus on infinite series? The easy answer that Newton was just dishonest does not capture the complexity of his position. Even if one wanted to concede dishonesty in Newton's handling of the quarrel with Leibniz, it would have been simply stupid not to provide evidence as strong as possible in *Commercium Epistolicum*.[19]

I believe that most of the perplexities that *Commercium Epistolicum* has engendered depend upon a lack of understanding of Newton's intentions and priorities. At this point, I therefore, devote some space to Newton's ideas on the nature of the calculus. What did Newton mean by *calculus* (or to use his terms, by the method of series and fluxions) in the context of the priority dispute? How did he address the issue of how the method had to be published and circulated? Indeed, the whole question of the priority in publishing the calculus cannot be broached without asking what terms like *calculus* and *publishing* meant for Newton.

As mentioned in chapter 16, printing was just one choice from a spectrum of means of communication available to him, which included manuscript circulation, oral communication, correspondence, and insertion of excerpts in books authored by other geometers. Newton believed that he had indeed already published his method of series and fluxions.

In *Commercium Epistolicum*, Newton also revealed that he had a different view of the nature and importance of the discovery that, he was convinced, Leibniz had stolen from him. I claim that while Leibniz insisted on the importance of his discovery and publication of the algorithm for differentiation, Newton and his acolytes focused on methods of quadrature (namely, integration) as the crucial issue.

Before commenting on *Commercium Epistolicum*, I briefly note that the context leading to its publication was polarized by matters concerning the inverse,

---

tum, et appropinquationes Arithmeticas transfert ad calculum Analyticum, sed nihil ad calculum differentialem. ... Cum ergo adversarii neque ex Commercio Epistolico, quod edidere, neque aliunde vel minimum indicium protulerint, unde constet aemulum tali calculo usum ante edita a nostro; ab his allata omnia ut aliena sperni possunt. Et usi sunt arte rabularum, ut judicantes a re de qua agitur ad alia diverterent, nempe ad series infinitas." Leibniz, *Mathematische Schriften* (1849–1863), 5, pp. 393, 410. See also Leibniz to Conti (March 29/April 9, 1716): "Lors que j'eus enfin le *Commercium Epistolicum*, je vîs qu'on s'y écartoit entirement du but, & que les Lettres qu'on publioit ne contenoient pas un mot qui peut faire revoquer en doute mon Invention du Calcul des Differences dont il s'agissoit. Au lieu de cela je remarquay qu'on se jettoit sur les Series." *Correspondence*, 6, pp. 305–6. This letter, now lost, is reproduced in Raphson, *History of Fluxions* (1715), pp. 103–11.

[19] For a well-balanced view of this question, see Hall, *Philosophers at War* (1980), pp. 188–9.

not the direct, method. Indeed, the priority dispute began in 1699 with an accusation of plagiarism addressed to Leibniz that was encapsulated in a work by Fatio on the brachistochrone, *Lineae Brevissimi Descensus Investigatio Geometrica.* Fatio was using fluxions in order to solve a difficult problem in the inverse method. The more proximate cause of the quarrel, Keill's paper on central forces (1708), was also devoted to the inverse method: the offending stab against Leibniz occurred in a paper devoted to a fluxional treatment of the inverse problem of central forces. This problem in mathematical dynamics, which was broached in terms of integration of differential equations, was to become a *cause célèbre* in the dispute between the Newtonians and the Leibnizians. When *Commercium Epistolicum* began to take shape in Newton's hands, the attention of mathematicians was caught by quadrature problems. The polemics between the brothers Bernoulli, or between Johann Bernoulli, Jacopo Riccati, and Jacob Hermann in the *Giornale dei Letterati*, were all concerned with inverse problems of quadrature. The challenges that Bernoulli delivered in 1696 were again on the inverse method (Methods 1, 2, and 3; see §8.4). Leibniz's insistence on the direct method, his pride in being the inventor of the algorithm for differentiation, appears unique. Perhaps he could share enthusiasm in researches on the direct method only with mathematicians like Varignon and L'Hospital, who were considered second-rank by the top winners of the French Academy prizes. Johann Bernoulli could easily sell the direct method to L'Hospital, but he was reluctant to pass his results on integration both to L'Hospital and to Varignon. The integral calculus was what he considered precious, and he jealously guarded it as his own province. It is thanks to his knowledge of integration that he established his reputation in the Malebranchiste circle in 1691; it is the discovery of integral calculus that he claimed as his own accomplishment in "Epistola pro Eminente Mathematico" (1716), addressed against Keill.

When Newton engineered his reply to Leibniz, it was natural for him to turn to the inverse method as the crucial issue. Indeed, *Commercium Epistolicum* begins in *Ad Lectorem* with the statement that the controversy relates to

> a general method of resolving finite equations into infinite ones, and applying these equations both finite and infinite, to the solution of problems by the proportions of the momentary augments of nascent and vanishing quantities.[20]

When one analyzes the mathematical examples adduced in *Commercium Epistolicum*, it emerges that Newton and his acolytes who were slavishly editing it

[20] "Newtonus ... habuit jam tum Methodum generalem aequationes finitas in infinitas resolvendi, & aequationes tum finitas tum infinitas applicandi ad Problemata solvenda, ope proportionum Augmentorum momentaneorum Quantitatum nascentium & augescentium." *Commercium Epistolicum* (1713), p. ii.

were referring to the inverse method of fluxions applied to problems of quadrature. This method consisted in expressing the ratio between fluxions (or, equivalently, moments) $\dot{y}/\dot{x}$ (in Leibnizian terms, between differentials) as an equation $f(\dot{y}, \dot{x}, x, y) = 0$, resolving the equation into an infinite series, and integrating term-wise (§7.4).

Here is a typical example. On page 15 of *Commercium Epistolicum* one finds a footnote to *De Analysi*—reproduced in its entirety roughly at the beginning of *Commercium Epistolicum*—which claims that what is presented is "an example of calculation by the moments of fluents."[21] This claim was an offending one for Leibniz.[22] On this page one finds a rectification of the circle's arc, namely, a quadrature, achieved via power series expansion and term-wise integration (see §7.4 and figure 7.5). The footnotes that mark, according to the editors of *Commercium Epistolicum*, examples of the method of fluxions are invariably referred to quadratures.

It is interesting to turn to the two *epistolae*, which together with *De Analysi* constitute the main evidence provided in *Commercium Epistolicum*. The two *epistolae* contain a great deal of information about quadrature techniques achieved via infinite series, and in the *epistola posterior* the higher quadrature techniques of the two catalogues of curves are also hinted at. In *Commercium Epistolicum* the two anagrams are deciphered and the second commented on by a long footnote that reveals Newton's priorities; it ends with a triumphant "no one doubts that Newton is the first discoverer of these rules." What is the claim of priority stated in this footnote?

> In his letters Newton reduced analysis by fluents and their momenta in equations, whether infinite or finite, to four rules. By the first a fluent is extracted from binomials, and so from any non-affected equations whatever by an infinite series, and at the same time it yields the moment of the fluent, at whose vanishing the series returns to a finite equation. [That is, the first rule is, expand the integrand via the binomial theorem and integrate term-wise; see §7.4.] By the second the fluent is extracted from affected equations not involving the fluxion. [That is, the second rule is, given a polynomial equation $f(x, y) = 0$, expand y as a fractional power series in x via analytical parallelogram and integrate term-wise; see §7.5.] By the third the fluent is extracted from affected equations which at the same time involve the fluxion. [That is, the third rule corresponds to Cases 1 and 2 of Problem 2 of *De Methodis*; see §8.4.2 and figure 8.11.] By the fourth the fluent is elicited from the conditions of the problem. [That is, the fourth rule is the method of undetermined coefficients.][23] The first two rules are put at the beginning of the former letter [the

[21] "Exemplum calculi per Momenta fluentium." *Commercium Epistolicum* (1713), p. 15 (p. 84, 2d edition).

[22] See the quotation from Leibniz's "Historia et Origo Calculi Differentialis" in this section.

[23] Doubts about the interpretation of the fourth rule are clarified by Newton in the "Account": "[H]e deduces converging Series from the Conditions of the Probleme, by assuming the Terms of the Series gradually, and determining them by those Conditions." Newton, "Account" (1715), p. 193.

> *epistola prior*], the last two at the end of this [the *epistola posterior*]. *No one doubts that Newton is the first discoverer of these rules.*[24]

Newton's rules were not the rules Leibniz was referring to in claiming his share in the discovery of the calculus. Leibniz was concerned with the direct method of differentiation, Newton with the inverse method of quadrature.

The fact that Newton and his acolytes focused on the use of series in the inverse method (integration) rather than on the direct method (differentiation) is supported by the attribution that one encounters repeatedly in their writings, of the direct method to Barrow, James Gregory, and Sluse rather than to Newton and Leibniz. Wallis, for instance, who was the first in 1693 to print the fluxional notation and algorithm in the second volume of his *Opera*, stated in his commentary that

> akin to this method [the direct method of fluxions] there is on the one hand the method of Leibniz, and on the other hand that method, older than either, which Dr Isaac Barrow has expounded in his *Lectiones Geometricae*: and this is acknowledged in the *Acta Lipsica* (Jan 1691).[25]

Acta Lipsica was *Acta Eruditorum* (published in Leipzig), and the reference is to a paper in which Jacob Bernoulli claimed that Leibniz's differential calculus was the same as Barrow's.[26] Of the same opinion was David Gregory, who in the opening lines of his *Isaaci Newtoni Methodus Fluxionum* made it clear that both Leibniz's and Newton's methods "slightly differ only in name" and "flow easily from Barrow's Method of Tangents treated in the 10th chapter of his *Lectiones Geometricae*."[27] Wallis and Gregory were simply repeating what Collins had reported to Wallis in the late 1670s.[28]

---

[24] "Analysin per Fluentes & earum Momenta in aequationibus tam infinitis quam finitis, Newtonus in his Epistolis ad regulas quatuor reduxit. Per primam extrahitur Fluens ex Binomiis, adeoque ex aequationibus quibuscunque non affectis in Serie infinita, & Momentum fluentis simul prodit, quo evanescente Series in Aequationem finitam redit. Per secundam extrahitur Fluens ex aequationibus affectis Fluxionem non involventibus. Per tertiam extrahitur Fluens ex aequationibus affectis Fluxionem simul involventibus. Per quartam eruitur Fluens ex conditionibus Problematis. Regulae duae primae in principio Epistolae superioris, duae ultimae in fine hujus ponuntur. Harum Regularum Newtonum esse inventorem primum nemo dubitat." *Commercium Epistolicum* (1713), p. 86. Emphasis supplied.

[25] "Huic Methodo affinis est tum Methodus differentialis Leibnitii, tum utraque antiquior illa quam D^r Is. Barrow in Lectionibus Geometricis exposuit. Quod agnitum est in *Actis Leipsiensis* (Anno 1691, mense Jan.) à quodam qui methodum adhibet Leibnitii similem." Wallis, *Opera*, 2, p. 396.

[26] See §8.1.4.

[27] Gregory refers also to Tschirnaus: "Calculus Differentialis Leibnizij et Methodus Fluxionum Newtoni, prior in Actis Lipsiae Octobris 1684, posterior a Wallisio Volumine altero Operum Mathematicorum 1692, tantum nomine tenus differant, ut et Tschirnausij Methodi Decembri 1682 et Martio 1683; quae omnes facile fluunt ex Methodo Tangentium Barrovij Lect: 10. Geom: tradita." Christ Church (Oxford), f. 1r.

[28] *Correspondence*, 2, pp. 242–3.

This interpretation seems not to have bothered Newton too much. Indeed, in *Commercium Epistolicum*, drafted under Newton's careful supervision, one repeatedly finds the direct method attributed to Barrow, James Gregory, and Sluse. In commenting on Leibniz's letter dated June 11/21, 1677, in which the rules of the differential calculus were first displayed to Newton, the editors of *Commercium Epistolicum* noted that "the very same thing achieved Barrow ... and by a very similar calculus."[29] Similarly, John Keill in his answer to Leibniz published in *Philosophical Transactions* for 1711, which was reproduced at the end of *Commercium Epistolicum*, stated that Sluse, James Gregory, and Barrow had methods for drawing tangents to curves "which do not differ too much from the method of fluxions."[30]

These statements, which appeared with Newton's approbation, have often been considered counterproductive. For instance, Montucla wrote,

> Is it not contradictory to say that Leibniz's method, described in the letter which we are considering [Leibniz to Oldenburg (June 11/21, 1677)], is just that of Barrow, and that it is the same as the one that Newton had communicated in 1669, which is claimed to be his method of fluxions? Because from this it follows that Newton's method is equivalent to Barrow's, excepting the notation.[31]

It is highly unlikely that Newton allowed the attribution of the direct method to Barrow because of careless editing of *Commercium Epistolicum*. One knows from the many manuscripts that Newton left that he supervised its publication with an almost obsessive attention to details of forensic relevance.[32]

---

[29] "Idem fecit D Barrow in ejus Lect. 10, Anno 1669 impressa, idque calculo consimili." *Commercium Epistolicum* (1713), p. 88.

[30] "Sciendum vero primum est, Celeberrimos tunc temporis Geometras, Dominos Franciscum Slusium, Isaacum Barrovium, & Jacobum Gregorium, Methodum habuisse qua Curvarum Tangentes ducebant, quae a Fluxionum methodo non multum abludebat." Keill opines that if one substitutes Gregory's symbol o, or Barrow's a and e with $\dot{x}$ and $\dot{y}$ or dx and dy one obtains Newton's and Leibniz's methods respectively. *Commercium Epistolicum* (1713), p. 112.

[31] "D'ailleurs n'y a-t-il pas de la contradiction à dire que la méthode de Leibnitz, décrite dans la lettre dont nous parlons [Leibniz to Oldenburg (June 11/21, 1677)], n'est que celle de Barrow, et qu'elle est la même que celle que Neuton [sic] avoit communiquée dès 1669, qu'on prétend être son calcul des fluxions. Car il suivroit delà que la méthode même de Neuton ne seroit que celle de Barrow à la notation près." Montucla, *Histoire des Mathématiques* (1799–1802), 3, p. 107. When Montucla died, pp. 1–336 of vol. 3 of his new edition of the *Histoire* had already been proofread and printed; the rest was revised by J. de Lalande (and Lalande availed himself of the help of several scholars; most notably S. F. Lacroix revised pp. 342–52 on integration of partial differential equations). See footnotes on pp. 336, 342, 344, 349 in vol. 3. Since volumes 3 and 4 of the *Histoire* are a cooperative effort, it is improper to attribute them to a single author. We do not know how heavily Montucla's text was changed, especially after p. 336. It is fair, I surmise, to attribute to Montucla quotations from pp. 1–336 of vol. 3.

[32] MP, 8, pp. 539–60.

Hall has an interesting explanation. According to his reading, one should interpret the pronouncements concerning Barrow's priority in the discovery of the direct method as an attempt to deflate Leibniz's challenge. Since it was difficult to gather evidence that Leibniz had obtained the direct method from Newton, because of Newton's secrecy on this point, it was functional to the maneuvers of the Royal Society's committee to state that the method was already at Leibniz's disposal in Barrow's *Lectiones Geometricae*, which Leibniz acquired while in London in 1673.[33] Further, in the Newtonian camp there must have been some awareness of the fact that Leibniz's algorithm for the direct (differential) method was superior to what Newton could offer, especially before the invention of the dotted notation in the 1690s. Newton in the 1670s (e.g, in *De Methodis*) had used letters such as n, m, l for the first fluxions of variable fluents x, y, z, and p, q, r for the second fluxions. Newton's early notation makes it unclear which fluxions belong to which variable. In *Commercium Epistolicum* and in Jones's 1711 edition of Newton's mathematical tracts, Newton's early manuscripts were edited to introduce the dotted notation, giving the impression it had been employed by Newton in his youth. Thus, according to Hall's analysis, attributing the direct method to Barrow rescued Newton from confronting Leibniz on a terrain that was unfavorable to Newton. There is certainly much truth in this interpretation, but I believe one should add that Newton, together with most of his contemporaries, showed little interest in the direct method for drawing tangents and in its algorithm, and was rather secretive and sensitive regarding his highly algebraic techniques in the inverse method (integration).

For modern readers the calculus is a deductive theory based on definitions (of limit, derivative, differential, etc.) and basic rules for differentiation. The crucial questions for modern interpreters have often been, Who was the first to discover these rules? and Who was the first to publish them? As a matter of fact, it is easier to find the rules of the differential calculus in Leibniz's "Nova Methodus" than in any of Newton's works. But these questions do not address what was crucial for Newton. A formal theory and its basic rules were not a matter of great interest for him. Rather, he was concerned with a method for resolving geometrical problems analytically via the method of moments and series, and he was, of course, much concerned also with the "splendid" geometrical constructions "worthy of public utterance." This method, in his opinion, showed its power only when tested against hard problems in squaring of curves or in the inverse method of tangents. Thus he saw himself as the discoverer not of simple rules for finding tangents but of a secret fountain that allowed him to solve such inverse problems.[34]

---

[33] Hall, *Philosophers at War* (1980), p. 55.

[34] There is an analogy here with Newton's attitude toward the three laws or axioms of motion of the *Principia*. Recent historiography on the *Principia* has put much stress on the laws. Questions such as, Was Newton indebted to Descartes or Galileo? Were his laws formulated in terms different from the ones accepted nowadays? When were Newton's laws applied in the eighteenth century?

The following remark from the third volume of Montucla's *Histoire* can be cited as an example of how, from a perspective different than Newton's, the *epistola posterior*, one of the chief evidential documents of *Commercium Epistolicum*, can be viewed as defective:

> Here we note that, after having read and re-read this letter [the *epistola posterior*], we find only the method of fluxions described as far as its consequences and advantages, but not as far as its principles.[35]

But Newton, unlike Leibniz, had always presented his analytical method of fluxions as a panoply of successful problem-solving techniques: its consequences and its advantages were his priority.

At this point, one can summarize the different positions held by the two combatants as follows. When evaluating the merits of his analytical method of fluxions over Leibniz's calculus, Newton focused on the rules for squaring curves of the inverse method. Leibniz instead based his claims as a discoverer of the calculus on the rules for the direct differential calculus. For Leibniz, too, the calculus was a heuristic tool, but he valued the systematic use of its symbolism and its logical structure highly. From his viewpoint, the rules of the direct differential calculus came first. By contrast, Newton never paid great attention to the rules of the direct method. For him the analytical method of fluxions was a patchwork of techniques that he did not attempt to systematize into a logical structure. The analytical method of fluxions was, in his opinion, in the end nondemonstrative. It was only its synthetic and geometrical form that could achieve mathematical certainty and that therefore needed a deductive structure whereby one posited legitimate postulates at its

---

Was Newton able to express $F = ma$? Did he understand the second law as valid for discontinuous impacts rather than for continuous forces? have occupied much attention and stimulated important research. But Newton did not consider the laws his most important contribution. He conceived the first two laws as belonging to the background knowledge of his days, attributing them to Galileo. His commentators rarely paid much attention to them. Typically, Gregory began his detailed commentary on the *Principia* ("Notae in Newtoni Principia Mathematica Philosophiae Naturalis," Royal Society, MS 210) with Section 1 on first and ultimate ratios. He did not waste a single word on commenting on the definitions, the laws, and their corollaries and scholia. Nowadays we identify Newton's mechanics with the three laws of motion and think about mechanics as a deductive axiomatic mathematical structure. And what can be more important than the axioms in an axiomatic structure? Newton and his acolytes instead understood the *Principia* as a treatise where mechanical problems are analytically resolved and synthetically constructed. Consequently their agenda and priorities differed from ours. The laws of motion were seen as expressing causal relations, the equivalent of artificial tracing mechanisms in organic geometry and the fluxional method, between forces and motions: the whole business of philosophy was applying analysis and synthesis to deduce motions from forces, and vice versa (see chapters 13 and 14).

[35] "Nous remarquons ici qu'après avoir lu et relu cette lettre, nous y trouvons seulement cette méthode [des fluxions] décrite, quant à ses effets et ses avantages, mais non quant à ses principes." Montucla, *Histoire des Mathématiques* (1799–1802), 3, p. 103.

beginning (see chapter 9). Consequently, Newton displayed the analytical method by successful applications to specific examples; he was convinced that its value lay in the fact that it permitted the resolution of hard quadrature problems. A typical statement that Newton anonymously circulated in 1717 can be cited as further evidence of his viewpoint:

> In the year 1684 Mr Leibnitz published only the Elements of the Calculus differentialis & applied them to questions about Tangents & Maxima & Minima as Fermat Gregory & Barrow had done before, & shewed how to proceed in these Questions without taking away surds, but proceeded not to the higher Problemes. The *Principia Mathematica* gave the first instances made publick of applying the Calculus to the higher Problemes.[36]

I do not detail here the intricate series of events that followed after the publication of *Commercium Epistolicum.* Most notably, Leibniz, in the "Charta Volans," his remarks on Keill's review of *Commercium Epistolicum* printed in French in *Journal Literaire*, and his letters to Conti was able to make his protestations known. Pressed by Leibniz's counterattacks, Newton and his acolytes had to abandon what appears to me the adamant coherence of *Commercium Epistolicum.* They had to explain that series were not extraneous to the issue, that Newton already had a notation and rules "not impeded by surds" for drawing tangents, and that he had an algorithm equivalent to the "véritable Calcul des différences." They had to explain why such an algorithm was not, in their opinion, so important; they had to justify many of their editorial choices behind the Royal Society's screed. It is due to Newton's and Leibniz's intellectual stature that these trivialities were soon replaced, or better intertwined, with a dialogue about a broad-ranging series of issues concerning the nature of mathematics, the status of geometry and algebra, the legitimacy of infinitesimals, and many more philosophical matters. These themes were touched on by Newton in the "Account" of *Commercium Epistolicum* that appeared anonymously in *Philosophical Translactions* for 1715.

17.3 The "Account"

It would be reductive to read the "Account" (1715) only as a polemicist essay. For my purposes, it is interesting to turn to some passages that reveal Newton's views on mathematical method in the most explicit way.

In the "Account," speaking of himself in the third person, Newton made clear that, in his opinion, Leibniz had only approached the analytical, heuristic part of the problem-solving method:

---

[36] Cambridge University Library, Add. 3968.41, f. 448r, cited in MP, 8, p. 513. This statement, written in 1716, appeared on page 117 of the anonymous appendix (pp. 111–9) to the 1717 reissue of Raphson, *The History of Fluxions* (1717), that Newton wrote.

> Mr. Newton's Method is also of greater Use and Certainty, being adapted either to the ready finding out of a Proposition by such Approximations as will create no Error in the Conclusion, or to the demonstrating it exactly; Mr. Leibnitz's is only for finding it out.[37]

So according to Newton, Leibniz had achieved only the first stage of the Pappian method and had not attained the exact, certain demonstration. The latter had to be carried out in purely geometrical terms (see chapter 9).

Further, Newton noted that in his method of first and ultimate ratios no infinitesimals occur, everything being performed according to limiting procedures. From Newton's point of view, the avoidance of infinitesimals and the possibility of interpreting algebraic symbols as geometrical magnitudes had the double advantage of endowing his method with referential content and being consonant with ancient mathematics:

> We have no Ideas of infinitely little Quantities & therefore Mr Newton introduced Fluxions into his Method that it might proceed by finite Quantities as much as possible. It is more Natural & Geometrical because founded upon *primae quantitatum nascentium rationes* [first ratios of nascent quantities] wch have a Being in Geometry, whilst *Indivisibles*, upon which the Differential Method is founded have no Being either in Geometry or in Nature. There are *rationes primae quantitatum nascentium*, but not *quantitates primae nascentes*. Nature generates Quantities by continual Flux or Increase, and the ancient Geometers admitted such a Generation of Areas and Solids .... But the summing up of Indivisibles to compose an Area or Solid was never yet admitted into Geometry.[38]

Nature and geometry are the two key concepts, allowing Newton to defend his synthetic method of fluxions because of its continuity with ancient tradition as well as because of its ontological content.

Finally, Newton insisted on the fact that the emphasis with which Leibniz praised the power of his symbolism was excessive. Algorithm is certainly important, but it has to be viewed only as a component of the method:

> Mr Newton – he wrote – doth not place his Method in Forms of Symbols, nor confine himself to any particular Sort of Symbols.[39]

Several notations are possible and equally acceptable. Newton was proud to show that he could frame his mathematics in different notations. Most notably, the drawing of tangents—the direct algorithm on which Leibniz insisted so much—could be achieved deploying different notations, or even without computation, making recourse to kinematic compositions of velocities (see §1.3 and figure 1.1). Newton

[37] Newton, "Account" (1715), p. 206.
[38] Newton, "Account" (1715), pp. 205–6.
[39] Newton, "Account" (1715), p. 204.

thought that the symbolical, algorithmic stage of analytical resolution had ultimately to be discarded from composition, and that only composition was worthy of being published (§4.5). Why then worry so much about a question of notation, as Leibniz did?

I conclude this chapter by pointing out, once more, that in these pronouncements one detects a tension and contradiction between Newton's deflationary statements concerning notation and his prodigiously fertile algebraic mathematical practice. Some of his mathematical achievements were possible thanks to his ability to trespass the boundaries of geometrical representability. This consideration applies especially to those results in the inverse method of fluxions that he so cherished. As Newton observed in the *epistola posterior* when comparing his squaring techniques with Leibniz's transmutation method,

> And indeed in the course of the progression [of the curves listed in the second catalogue; see §8.4.4] all soon became very complicated, so that I hardly think they can be found by the transformation of the figures, which Gregory and others have used [a reference to the method of transmutation delivered in Leibniz's reply to the *epistola prior*; see §16.3.5, which Newton compared with the geometrical quadratures of James Gregory], without some further foundation. Indeed I myself could gain nothing at all general in this subject before I withdrew from the contemplation of figures and reduced the whole matter to the simple consideration of ordinates alone.[40]

The "simple consideration of ordinates" is, of course, the technique of variable substitution that allowed Newton to construct his second catalogue of curves, where he reduced the squaring of a curve whose ordinate is y and abscissa z to the squaring of a conic whose ordinate is v and abscissa x. This defense of the power of symbolism in freeing the mind from the "contemplation of figures" is almost unique in Newton's writings yet is revealing of the gap between his views on mathematical method and his mathematical practice. Indeed, the preceding statement is a criticism of Leibniz's method, which, being based on a geometrical transmutation of geometrical infinitesimals surface components, is deemed less powerful compared with the purely formal substitutions of variables that characterize Newton's quadrature techniques. That is, while in Leibniz's method the curve is squared by decomposing the surface bounded by it into infinitesimal components, Newton, in order to achieve the quadratures of the second catalogue, considered the algebraic relation between abscissa and ordinate, abstracting from geometrical representation.[41]

---

[40] *Correspondence*, 2, p. 138.

[41] See also Newton's comments in his letter to Collins dated November 8, 1676: "'As for ye apprehension yt M. Leibnitz's method may be more general or more easy then mine, you will not find any such thing ... As for ye method of Transmutation in general, I presume he has made further improvements then others have done, but I dare say all that can be done by it may be done better wthout it, by ye simple consideration of the ordinatim applicatae." *Correspondence*, 2, p. 179.

The "Account" ends with a long paragraph opened by the words, "[I]t must be allowed that these two Gentlemen differ very much in Philosophy."[42] Newton proceeded to succinctly list a series of philosophical points of disagreement between himself and Leibniz, from the role of experiments to the cause of gravity, from the nature of miracles to the power of God on natural phenomena. The mathematical controversy expanded in a complex philosophical confrontation that culminated in the correspondence between Leibniz and Samuel Clarke.[43]

[42] Newton, "Account" (1715), p. 224.
[43] See Hall, "Newton versus Leibniz" (2002).

Conclusion

After this detailed analysis of Newton's writings on mathematical method it is time to briefly evaluate what has been achieved. Does Newton emerge as a creative and innovative philosopher of mathematics? Clearly, the answer is no. From this viewpoint, he does not compare with Descartes and Leibniz. His methodological views were framed in terms that are commonplace in the history of seventeenth-century mathematics. Newton's conceptions on analysis and synthesis, or on the merits of geometry over algebra, were shared by many of his contemporaries, including Hobbes, Fermat, Huygens, and Barrow. Descartes and Leibniz, would have subscribed to many of Newton's statements concerning the certainty of geometry and the concealed analysis of the ancients.[1] Devoting attention to commonplaces is, however, rewarding for a historian, since commonplaces can be put into use for very different purposes by different historical actors. As I have tried to show, these commonplaces meant something particular to Newton that he was at pains to enforce in his milieu. It is my fascination with the idiosyncratic peculiarity of Newton's views on mathematical method that motivated this book.

Eminently Newton was a mathematician, and like many great mathematicians he was an innovator, an opportunist, and careless about rigor in his forays into virgin territory. His mathematical practice lacks systematicity, it appears as a patchwork of problem-solving techniques that is difficult to schematize. When engaged in solving a problem, he could make recourse to his knowledge of Apollonian geometry, to Wallisian inductions, or even to instruments, such as rotating rulers, that he manipulated. In some instances Newton relied upon unproven statements, which turned out to be deep truths that he just glimpsed. Such lack of systematicity is often caused by the fertility of his mind, by the fact that he excelled in mathematical fields so far apart one from the other. He could produce interesting results in infinite series as well as in projective geometry, in the theory of equations as well as in mathematical physics.

Newton's unsystematic but efficient mathematical practice was at odds with his philosophical agenda. He sought certainty in mathematics, and never ceased to see mathematics as the vehicle for delivering certainty in natural philosophy. He was opposed to the anticlassical stance that he perceived in Descartes' *Géométrie* and portrayed himself as indebted to Euclid and Apollonius rather than to the moderns.

[1] See, e.g., Descartes, *Meditationes de Prima Philosophia* (1641) (AT, 7, pp. 155–6); Leibniz's review of *Arithmetica Universalis* in *Acta Eruditorum* (November 1708), pp. 519–26, reproduced in MP, 5, pp. 23–31 (on p. 30).

This notwithstanding, his mathematical methods are a Cartesian heritage. He tried therefore to reformulate his analytical methods of discovery into a synthetic form, a form in which all reference to algebraic analysis is suppressed—the equation is neglected—and the purity, unity, and beauty of geometry recovered. This was largely a failure.

The fact is that the symbolical tools in the common and new analyses that Newton mastered so well were becoming increasingly autonomous from geometry. When he turned algebra into synthetic constructions, he relied on aesthetic criteria of exactness and simplicity that were arbitrary and unjustified (see part II). When he reformulated the new analysis into the synthetic method of fluxions he could not capture the eminently symbolical character of his quadrature techniques (see part III). As he wrote to Leibniz, he could achieve his quadratures by "withdrawing from the contemplation of the figures." His reticence in making explicit all the quadrature techniques in the *Principia* rendered some portions of his masterpiece unintelligible exactly because the geometrical constructions of the results achieved via the inverse method of fluxions could not—contrary to what he maintained—reveal the analysis even to very skillful mathematicians (see part IV). The geometrical constructions that Descartes and Newton conceived were remote from the analytical symbolic process of discovery. Newton was keenly aware of this asymmetry between modern algebraic analysis and geometrical synthesis and tried hard to recover and extend the analytical geometrical process of discovery to curvilinear figures. But this did not lead him to recover the unity between analysis and synthesis that, he was convinced, characterized the mathematical practices of the ancients. What he obtained was a series of interesting results in projective geometry that never reached systematicity in his hands. Newton's deployment of the analysis/synthesis dichotomy was basic to his reflections on the role of mathematics in the mathematization of natural phenomena. But here also he did not encounter success; he was able to formulate some rhetorical pronouncements on the method of analysis and synthesis in natural philosophy that do not stand up to close scrutiny (see part V).

I have tried to sketch a diachronic image of Newton the philosopher of mathematics. This book has followed him from his early unsystematic inventive researches carried on in the 1660s and his bold programmatic statement on the role of mathematics in natural philosophy (1670), to his more mature lucubrations in the mid-1670s, when as Lucasian Professor he began to study ancient geometry and to position himself against Cartesian mathematics. It has delved into the tensions that characterized the *Principia* (1687) and into the period of reformulations that followed its publication, when Newton mused on the myth of the ancients' wisdom. It has examined how he modified his publication strategy because of the changes in the mathematical scene in England and on the Continent, most notably because of Leibniz's entrance into the arena. This development, this tortuous and suffered trajectory, also has something to tell about Newton, about the seriousness with which he faced the problem of mathematical certainty and about the failures he experienced.

The lack of originality and aporetic character of Newton's views on mathematical method, and the tensions that these views created with the mathematical methods that he practiced, are evident. And perhaps these drawbacks and failures are the reason this fragment of Newton's work has been ignored so far. But this fragment is important for understanding significant aspects of Newton's intellectual biography as it interacted with his views on the ancients and moderns, on Cartesianism, and on the role of mathematics in natural philosophy. This fragment also determined Newton's approach to publication, shaped his relationships with his acolytes, and influenced his strategy in the polemic with Leibniz. I hope that this book, by looking at these aspects of Newton's thought and work, will be of some help to Newtonian scholarship.

A Brief Chronology of Newton's Mathematical Work

1661	enters Trinity College, Cambridge
1664?	fruitful relationship with Barrow begins shaping his mathematical ideas
1664–1665	notes on Wallis's *Arithmetica Infinitorum* (1655) lead to binomial series
1665–1666	*anni mirabiles* culminate in the October 1666 Tract on Fluxions
1669	work on infinite series applied to quadratures in *De Analysi* sent to Collins
1669	appointed Lucasian Chair
1670–1671	work on analytical method of fluxions in *De Methodis* left unfinished
1670–1672	in *Optical Lectures* the role of geometry in natural philosophy is defended
1672	elected Fellow of the Royal Society (January 1/11, 1671/72)
1672	the polemic on "New Theory about Light and Colors" begins
1673	Newton receives a copy of Huygens's *Horologium Oscillatorium*
1670s (mid)	work on interpolation
1676	*epistola prior* and *epistola posterior* sent to Leibniz
1670s (late)	Pappus problem solved in "Solutio Problematis Veterum de Loco Solido"
1670s (late)	"Errores Cartesij Geometriae": criticisms against Descartes
1670s (late)	work on classification of cubics
1679–1680	correspondence with Hooke on planetary motions
1680?	work on synthetic method of fluxions in "Geometria Curvilinea"
1683–1684	*Lucasian Lectures on Algebra* deposited
1684	Halley visits Newton; "De Motu Corporum" sent to Royal Society
1684–1686	composition of *Principia*
1685	binomial series and quadratures via infinite series in Wallis's *Algebra*
1685	Craig visits Newton and receives the prime theorem
1687	publication of *Principia*
1691–1692	composition of *De Quadratura*
1690s (early)	work on the restoration of *Porisms* in "Geometriae Libri Duo"
1693	fluxional notation and quadrature methods in vol. 2 of Wallis's *Opera*
1694	Gregory visits Newton; receives information on Newton's mathematics and a letter on the use of quadratures in *Principia*
1690s (mid)	evidence of Newton's endorsement of the myth of a *prisca sapientia*
1690s (mid)	Newton-Cotes formula in "Of Quadrature by Ordinates"
1695	composition of *Enumeratio*
1696	Newton moves to London as Warden of the Mint
1699	Fatio's *Lineae Brevissimi Descensus*; Leibniz accused of plagiarism
1699	full text of the two *epistolae* for Leibniz in vol. 3 of Wallis's *Opera*
1703	elected President of the Royal Society
1704	*De Quadratura* and *Enumeratio* published in appendix to *Opticks*
1705	Leibniz's anonymous review of *De Quadratura*
1706	Quaestio 23 in *Optice* on analysis and synthesis in natural philosophy
1707	Whiston supervises the publication of *Arithmetica Universalis*

1710	Leibniz accused of plagiarism in Keill's "Epistola" (1708)
1711	*De Analysi*, *Methodus Differentialis*, *De Quadratura*, and *Enumeratio* published by Jones in Newton, *Analysis per Quantitatum* (1711)
1711 (late)	Leibniz writes to Sloane asking for Keill's apology
1713 (early)	*Commercium Epistolicum* distributed free of cost
1713	second edition of *Principia*
1715	anonymous "Account" of *Commercium Epistolicum*
1726	third edition of *Principia*

References

Acerbi, Fabio. "Introduzione." In *Euclide, Tutte le Opere*, edited and translated by Fabio Acerbi, pp. 15–774. Milan: Bompiani, 2007.

Adams, John C. "On Newton's Solution of Kepler's Problem." *Monthly Notices of the Royal Astronomical Society* 43 (1882): 43–9.

Adams, John C., George D. Liveing, Henry R. Luard, and George G. Stokes. *A Catalogue of the Portsmouth Collection of Books and Papers Written by or Belonging to Sir Isaac Newton: The Scientific Portion of Which Has Been Presented by the Earl of Portsmouth to the University of Cambridge, Drawn up by the Syndicate Appointed.* Cambridge: Cambridge University Press, 1888.

Albers, Donald J. "Freeman Dyson: Mathematician, Physicist, and Writer." *College Mathematics Journal* 25, no. 1 (January 1994): 2–21. (Albers's interview with Dyson).

Andersen, Kirsti. "Cavalieri's Method of Indivisibles." *Archive for History of Exact Sciences* 31 (1985): 291–367.

—. "Newton's Inventive Use of Kinematics in Developing His Method of Fluxions." *Oberwolfach Reports* 2, no. 4 (2005): 3188–93.

Apollonius. *De Sectione Rationis Libri Duo.* Edited by Edmond Halley. Oxford: E Theatro Sheldoniano, 1706.

—. *Apollonii Pergaei Conicorum Libri Octo et Sereni Antissensis De Sectione Cylindri & Coni Libri Duo.* Edited by Edmond Halley. Oxford: E Theatro Sheldoniano, 1710.

Arthur, Richard T. W. "Newton's Fluxions and Equably Flowing Time." *Studies in History and Philosophy of Science* 26 (1995): 323–51.

Bailly, Jean S. *Histoire de l'Astronomie Moderne: Depuis la Fondation de l'École d'Alexandrie Jusqu'à l'Époque de 1730.* 3 vols. Paris: de Bure, 1785.

Barbin, Evelyne, Jacques Borowczyk, Jean-Luc Chabert, Michel Guillemot, and Anne Michel-Pajus. *A History of Algorithms: From the Pebble to the Microchip.* Berlin: Springer, 1999.

Barrow, Isaac. *Lectiones Geometricae: In Quibus (Praesertim) Generalia Curvarum Linearum Symptomata Declarantur.* London: Godbid, 1670.

—. *Lectiones Opticae & Geometricae: In Quibus Phaenomenωn Opticorum Genuinae Rationes Investigantur, ac Exponuntur: Et Generalia Curvarum Linearum Symptomata Declarantur.* London: Godbid, 1674.

—. *Archimedis Opera: Apollonii Pergaei Conicorum libri IIII, Theodosii Sphaerica, Methodo Nova Illustrata & Succincte Demonstrata per Is. Barrow ... Accedunt Ejusdem Lectiones Opticae, & Geometricae.* London: Godbid, 1675.

—. *Lectiones Mathematicae XXIII. In Quibus Principia Matheseωs Generalia Exponuntur.* London: Typis J. Playford, 1683.

—. *The Usefulness of Mathematical Learning Explained and Demonstrated: Being Mathematical Lectures Read in the Publick Schools at the University of Cambridge.* Translated by John Kirkby. London: for Stephen Austen, 1734.

—. *Geometrical Lectures: Explaining the Generation, Nature and Properties of Curve Lines.* Edited by Edmund Stone. London: for Stephen Austen, 1735.

—. *The Mathematical Works of Isaac Barrow.* Edited by William Whewell. Cambridge: Cambridge University Press, 1860.

—. *The Geometrical Lectures of Isaac Barrow: Translated, with Notes and Proofs, and a Discussion on the Advance Made Therein of the Work of his Predecessors in the Infinitesimal Calculus.* Translation and notes by James M. Child. London: Open Court, 1916.

Bartolazzi, Margherita, and Raffaella Franci. "Un Frammento di Storia dell'Algebra: La Regola di Newton sul Numero di Radici Immaginarie di un'Equazione Algebrica." *Rendiconti dell'Accademia Nazionale delle Scienze Detta dei XL* 14 (1990): 69–85.

Beal, Peter. *In Praise of Scribes: Manuscripts and Their Makers in Seventeenth-Century England.* Oxford: Clarendon Press, 1998.

Bechler, Zev. "Newton's 1672 Optical Controversies: A Study in the Grammar of Scientific Dissent." In *The Interaction Between Science and Philosophy*, edited by Yehuda Elkana, pp. 115–42. Atlantic Highlands, N.J.: Humanities Press, 1974.

Bennett, Jim. *The Mathematical Science of Christopher Wren.* Cambridge: Cambridge University Press, 1982.

—. "The Mechanics' Philosophy and the Mechanical Philosophy." *History of Science* 24 (1986): 1–28.

Berkeley, George. *The Analyst: Or, a Discourse Addressed to an Infidel Mathematician. Wherein it is Examined Whether the Object, Principles, and Inferences of the Modern Analysts are More Distinctly Conceived, or More Evidently Deduced, than Religious Mysteries and Points of Faith, by the Author of the Minute Philosopher.* London: for J. Tonson, 1734.

[Bernoulli, Johann]. "Epistola pro Eminente Mathematico, Dn. Johanne Bernoullio, Contra Quendam ex Anglia Antogonistam [sic] Scripta." *Acta Eruditorum* (Julii 1716): 296–315.

Bertoloni Meli, Domenico. *Equivalence and Priority: Newton versus Leibniz.* Oxford: Clarendon Press, 1993.

—. *Thinking with Objects: The Transformation of Mechanics in the Seventeenth Century.* Baltimore: Johns Hopkins University Press, 2006.

Birch, Thomas. *The History of the Royal Society of London for Improving of Natural Knowledge from its First Rise: In Which the Most Considerable of Those Papers Communicated to the Society, Which Have Hitherto Not Been Published, Are Inserted in Their Proper Order, as a Supplement to the Philosophical Transactions.* London: for A. Millar, 1756–57.

Blay, Michel. *La Naissance de la Mécanique Analytique: La Science du Mouvement au Tournant des XVIIe et XVIIIe Siècles*. Paris: Presses Universitaires de France, 1992.

—. *Reasoning with the Infinite: From the Closed World to the Mathematical Universe*. Chicago: University of Chicago Press, 1998.

Boas, Mary. *Robert Boyle and Seventeenth-Century Chemistry*. Cambridge: Cambridge University Press, 1958.

Bos, Henk J. M. "On the Representation of Curves in Descartes' *Géométrie*." *Archive for History of Exact Sciences* 24 (1981): 295–338.

—. "Arguments on Motivation in the Rise and Decline of a Mathematical Theory: The 'Construction of Equations,' 1637–ca. 1750." *Archive for History of Exact Sciences* 30 (1984): 331–80.

—. *Redefining Geometrical Exactness: Descartes' Transformation of the Early Modern Concept of Construction*. New York: Springer, 2001.

Boyle, Robert. *Hydrostatical Paradoxes: Made Out by New Experiments, (for the Most Part Physical and Easie)*. Oxford: W. Hall, for R. Davis, 1666. In Robert Boyle, *The Works of the Honourable Robert Boyle in Five Volumes*. Vol. 2, pp. 738–97. London: for A. Millar, 1744.

—. *Of the Usefulness of Mathematicks to Natural Philosophy: Or, That the Empire of Man May be Promoted by the Naturalist's Skill in Mathematicks, (as well Pure, as Mixed)*. Oxford: H. Hall, for R. Davis, 1671. In Robert Boyle, *The Works of the Honourable Robert Boyle in Five Volumes*. Vol. 3, pp. 392–456. London: for A. Millar, 1744.

—. *Medicina Hydrostatica, or, Hydrostaticks Applyed to the Materia Medica: Shewing How by the Weight that Divers Bodies, Us'd in Physick, Have in Water: One May Discover Whether They Be Genuine or Adulterate, To Which is Subjoyn'd a Previous Hydrostatical Way of Estimating Ores*. London: for Samuel Smith, 1690. In Robert Boyle, *The Works of the Honourable Robert Boyle in Five Volumes*. Vol. 5, pp. 453–89. London: for A. Millar, 1744.

—. *The Works of the Honourable Robert Boyle in Five Volumes: To Which is Prefixed the Life of the Author*. London: for A. Millar, 1744.

—. *The Works of Robert Boyle*. Edited by Michael Hunter and Edward B. Davis. 14 vols. London: Pickering and Chatto, 1999–2000.

Brackenridge, Bruce. *The Key to Newton's Dynamics*. Berkeley: California University Press, 1995.

—. "Newton's Easy Quadratures: 'Omitted for the Sake of Brevity'." *Archive for History of Exact Sciences* 57 (2003): 313–36.

Brackenridge, Bruce, and Michael Nauenberg. "Curvature in Newton's Dynamics." In *The Cambridge Companion to Newton*, edited by I. Bernard Cohen and George E. Smith, pp. 85–137. Cambridge: Cambridge University Press, 2002.

Brieskorn, Egbert, and Horst Knörrer. *Plane Algebraic Curves*. Basel: Birkhäuser, 1986.

Brigaglia, Aldo. "La Riscoperta dell'Analisi e i Problemi Apolloniani." In *Geometria, Flussioni, Differenziali: Tradizione e Innovazione nella Matematica del Seicento*, edited by Marco Panza and Clara Silvia Roero, pp. 221–69. Naples: La Città del Sole, 1995.

Buchwald, Jed Z. "Descartes's Experimental Journey Past the Prism and Through the Invisible World to the Rainbow." *Annals of Science* 65 (2008): 1–46.

Cantor, Moritz. *Vorlesungen über Geschichte der Mathematik.* Vol. 3. 2d ed. Leipzig: B.G. Teubner, 1901.

Casey, John. *A Treatise on the Analytical Geometry of the Point, Line, Circle, and Conic Sections: Containing an Account of Its Most Recent Extensions, with Numerous Examples.* Dublin: Hodges, Figgis, 1893.

Chandrasekhar, Subrahmanyan. *Newton's Principia for the Common Reader.* Oxford: Clarendon Press, 1995.

Chasles, Michel. *Les Trois Livres de Porismes d'Euclide.* Paris: Mallet-Bachelier, 1860.

Cheyne, George. *Fluxionum Methodus Inversa: Sive Quantitatum Fluentium Leges Generaliores.* London: J. Matthew, for R. Smith, 1703.

Cohen, I. Bernard. *Introduction to Newton's Principia.* Cambridge: Cambridge University Press, 1971.

—. *Isaac Newton's Theory of the Moon's Motion, 1702: With a Bibliographical and Historical Introduction.* Folkestone, U.K.: Dawson, 1975. Reprint of Newton, *A New and Most Accurate Theory of the Moon's Motion* (1702).

—. *The Newtonian Revolution.* Cambridge: Cambridge University Press, 1980.

—. "A Guide to Newton's *Principia.*" In Isaac Newton, *The Principia: Mathematical Principles of Natural Philosophy*, translated by I. Bernard Cohen and Anne Whitman, assisted by Julia Budenz, pp. 1–370. Berkeley: University of California Press, 1999.

Commercium Epistolicum D. Johannis Collins, et Aliorum De Analysi Promota: Jussu Societatis Regiae In Lucem Editum. London: typis Pearsonianis, 1712. Printed late 1712 but distributed January/February 1713 (N.S.).

Commercium Epistolicum D. Johannis Collins, et Aliorum, De Analysi Promota, Jussu Societatis Regiae in Lucem Editum et Jam Una Cum Ejusdem Recensione Praemissa & Judicio Primarii, ut Ferebatur, Mathematici Subjuncto, Iterum Impressum. London: J. Tonson and J. Watts, 1722.

Commercium Epistolicum J. Collins et Aliorum de Analysi Promota: Ou Correspondance de J. Collins et d'Autres Savants Célèbres du XVII Siècle, Relative à l'Analyse Supérieure. Edited by Jean-Baptiste Biot and Felix Lefort. Paris: Mallet-Bachelier, 1856.

Coolidge, Julian Lowell. *A History of Geometrical Methods.* Oxford: Clarendon Press, 1940. Edition cited: New York: Dover, 2003.

Copernicus, Nicolaus (Kopernik). *De Revolutionibvs Orbium Coelestium, Libri VI.* Nuremberg: Apud J. Petreium, 1543.

Craig, John. *Methodus Figurarum Lineis Rectis & Curvis Comprehensarum Quadraturas Determinandi.* London: M. Pitt, 1685.

—. *Tractatus Mathematicus de Figurarum Curvilinearum Quadraturis et Locis Geometricis.* London: S. Smith and B. Walford, 1693.

Cramer, Gabriel. *Introduction à l'Analyse des Lignes Courbes Algébriques.* Geneva: Chez les Frères Cramer and Cl. Philibert, 1750.

Cuomo, Serafina. *Pappus of Alexandria and the Mathematics of Late Antiquity*. Cambridge: Cambridge University Press, 2000.

De Gandt, François. "Le Style Mathématique des *Principia* de Newton." *Revue d'Histoire des Sciences* 39 (1986): 195–222.

—. *Force and Geometry in Newton's Principia*. Princeton: Princeton University Press, 1995.

De Moivre, Abraham. *Animadversiones in D. Georgii Cheynaei Tractatum de Fluxionum Methodo Inversa*. London: E. Midwinter, for D. Midwinter and T. Leigh, 1704.

De Morgan, Augustus. "On the Additions Made to the Second Edition of the *Commercium Epistolicum*." *Philosophical Magazine* 23 (1848): 446–56.

—. "A Short Account of Some Recent Discoveries in England and Germany Relating to the Controversy on the Invention of Fluxions." *Companion to the Almanac* (1852): 5–20. Reprinted in Augustus De Morgan, *Essays on the Life and Work of Newton*, pp. 67–101. London: Open Court, 1914.

—. "On the Authorship of the Account of the *Commercium Epistolicum*." *Philosophical Magazine* 4 (1852): 440–4.

Dear, Peter. *Discipline and Experience: The Mathematical Way in the Scientific Revolution*. Chicago: University of Chicago Press, 1995.

Dechales, Claude-François M. *Cursus seu Mundus Mathematicus*. Lyon: Ex Officina Anissoniana, 1674.

Densmore, Dana. *Newton's Principia, the Central Argument: Translation, Notes, and Expanded Proofs*. Translation and illustrations by William H. Donahue. Santa Fe, N.M.: Green Lion Press, 1995. 3d rev ed, 2003.

Descartes, René. *La Géométrie*. In René Descartes, *Discours de la Méthode, pour Bien Conduire sa Raison et Chercher la Verité dans les Sciences: Plus la Dioptrique, Les Meteores, et la Géométrie, Qui Sont des Essais de Cete Methode*, pp. 297–413. Leiden: Maire, 1637. Edition cited: *The Geometry of René Descartes with a Facsimile of the First Edition*. Edited and translated by D. E. Smith and M. L. Latham. New York: Dover, 1954.

—. *Principia Philosophiae*. Amsterdam: Apud Ludovicum Elzevirium, 1644.

—. *Les Passions de l'Âme*. Paris: Chez H. Le Gras, 1649. Edition cited: *Discours sur la Méthode—Méditations Philosophiques—Les Passions de l'Âme—Règles pour la Direction de l'Esprit—De la Vérité par les Lumières Naturelles*. Edited by Louis Aimé-Martin. Paris: Lefévre, 1844.

—. *Geometria, à Renato Des Cartes Anno 1637 Gallicè Edita*. Leiden: Ex Officina J. Maire, 1649.

—. *Opera Philosophica*. 3d ed. Amsterdam: Apud Ludovicum et Danielem Elzevirios, 1656.

—. *Geometria, à Renato Des Cartes Anno 1637 Gallicè Edita*. Amsterdam: Apud Ludovicum et Danielem Elzevirios, 1659–61.

—. *Oeuvres de Descartes*. Edited by Charles Adam and Paul Tannery. 11 vols. New ed. Paris: Vrin, 1964–67. First edition Paris: Cerf, 1897–1913.

Di Sieno, Simonetta, and Massimo Galuzzi. "La Quinta Sezione del Primo Libro dei *Principia*: Newton e il Problema di Pappo." *Archives Internationales d'Histoire des Sciences* 39 (1989): 51–68.

DiSalle, Robert. *Understanding Spacetime: The Philosophical Development of Physics from Newton to Einstein*. Cambridge: Cambridge University Press, 2006.

Ditton, Humphry. *The General Laws of Nature and Motion; with their Application to Mechanicks: Also the Doctrine of Centripetal Forces, and Velocities of Bodies, Describing any of the Conick Sections, Being a Part of the Great Mr. Newton's Principles, The Whole Illustrated with Variety of Useful Theorems and Problems, and Accomodated to the Use of the Younger Mathematicians*. London: T. Mead, for J. Seller, C. Price, and J. Senex, 1705.

Domski, Mary. "The Constructible and the Intelligible in Newton's Philosophy of Geometry." *Philosophy of Science* 70 (2003): 1114–24.

Ducheyne, Steffen. "Newton's Training in the Aristotelian Textbook Tradition: From Effects to Causes and Back." *History of Science* 43 (2005): 217–37

Dupont, Pascal. Appunti di Storia di Analisi Infinitesimale. Lecture notes of the course on the history of mathematics held at the Università degli Studi di Torino in 1981–82.

Eagles, Christina M. The Mathematical Work of David Gregory, 1659–1708. Ph.D. diss., University of Edinburgh, 1977.

Edleston, Joseph. *Correspondence of Sir Isaac Newton and Professor Cotes*. London: J. W. Parker, 1850.

Edwards, Charles H. *The Historical Development of the Calculus*. New York: Springer, 1979.

Erlichson, Herman. "The Visualization of Quadratures in the Mystery of Corollary 3 to Proposition 41 of Newton's *Principia*." *Historia Mathematica* 21 (1994): 148–61.

Euclid. *Elementorum Libri XV Breviter Demonstrati: Opera Is. Barrow Cantabrigiensis*. Cambridge: W. Nealand, 1655.

Euler, Leonhard. *Mechanica: Sive, Motus Scientia Analytice Exposita*. Saint Petersburg, 1736. In *Opera Omnia*, 2d ser., Vols. 1–2.

Fatio de Duillier, Nicolas. *Lineae Brevissimi Descensus Investigatio Geometrica Duplex: Cui Addita Est Investigatio Geometrica Solidi Rotundi in Quod Minima Fiat Resistentia*. London: R. Everingham, for J. Taylor, 1699.

Feingold, Mordechai. *The Mathematicians' Apprenticeship: Science, Universities and Society in England, 1560–1640*. Cambridge: Cambridge University Press, 1984.

—. "Isaac Barrow: Divine, Scholar, Mathematician." In *Before Newton: The Life and Times of Isaac Barrow*, edited by Mordechai Feingold, pp. 1–104. Cambridge: Cambridge University Press, 1990.

—. "Isaac Barrow's Library." In *Before Newton: the Life and Times of Isaac Barrow*, edited by Mordechai Feingold, pp. 333–72. Cambridge: Cambridge University Press, 1990.

—. "Newton, Leibniz, and Barrow Too: An Attempt at a Reinterpretation." *Isis* 84 (1993): 310–38.

—. "Mathematicians and Naturalists: Sir Isaac Newton and the Royal Society." In *Isaac Newton's Natural Philosophy*, edited by Jed Z. Buchwald and I. Bernard Cohen, pp. 77–102. Cambridge, Mass.: MIT Press, 2001.

Fermat, Pierre de. *Varia Opera Mathematica*. Toulouse: Apud Joannem Pech, 1679.

Figala, Karin. "Die Exakte Alchemie von Isaac Newton." *Verhandlungen der Naturforschenden Gesellschaft in Basel* 94 (1984): 157–228.

Fleckenstein, Joachim O. "Johann I Bernoulli als Kritiker der *Principia* Newtons." *Elemente der Mathematik* 1, no. 6 (1946): 100–8.

—. "Der Prioritätstreit zwischen Leibniz und Newton: Isaac Newton." *Elemente der Mathematik* Beihefte 12 (1956): 1–27. 2d ed. Basel: Birkhäuser, 1977.

Fraser, Duncan C. *Newton's Interpolation Formulas*. London: C. and E. Layton, 1927.

Frost, Percival. *An Elementary Treatise on Curve Tracing*. 5th ed. New York: Dover, 2004. First published London: Macmillan, 1872.

Funkenstein, Amos. *Theology and the Scientific Imagination from the Middle Ages to the Seventeenth Century*. Princeton: Princeton University Press, 1986.

Gabbey, Alan. "Newton's *Mathematical Principles of Natural Philosophy*: A Treatise on 'Mechanics'?" In *The Investigation of Difficult Things: Essays on Newton and the Exact Sciences*, edited by Peter M. Harman and Alan E. Shapiro, pp. 305–22. Cambridge: Cambridge University Press, 1992.

Galilei, Galileo. *Mathematical Discourses and Demonstrations Touching Two New Sciences, Pertaining to the Mechanicks and Local Motion*. In Thomas Salusbury, *Mathematical Collections and Translations in Two Tomes (Englished from the Originall Latine and Italian*. 2 vols. London: W. Leybourn, 1661–65.

—. *Dialogo Sopra i Due Massimi Sistemi del Mondo, Tolemaico e Copernicano*. Edited by Ottavio Besomi and Mario Helbing. Padua: Antenore, 1998.

Galuzzi, Massimo. "Il Problema delle Tangenti nella 'Géométrie' di Descartes." *Archive for History of Exact Sciences* 22 (1980): 37–51.

—. "I Marginalia di Newton alla Seconda Edizione Latina della Geometria di Descartes e i Problemi ad Essi Collegati." In *Descartes, il Metodo e i Saggi: Atti per il 350º Anniversario della Pubblicazione del Discours de la Méthode e degli Essais*, edited by Giulia Belgioioso, Guido Cimino, Pierre Costabel, and Giovanni Papuli, pp. 387–417. Rome: Istituto dell'Enciclopedia Italiana, 1990.

Galuzzi, Massimo, and Daniela Rovelli. *Nouveauté et Modernité dans les Mathématiques de Descartes*. Cosenza: Rubbettino, forthcoming.

Garber, Daniel. *Descartes' Metaphysical Physics*. Chicago: University of Chicago Press, 1992.

Garrison, James W. "Newton and the Relation of Mathematics to Natural Philosophy." *Journal of the History of Ideas* 48 (1987): 609–27.

Gaukroger, Stephen. *Descartes' System of Natural Philosophy*. Cambridge: Cambridge University Press, 2002.

Giusti, Enrico. *Bonaventura Cavalieri and the Theory of Indivisibles*. Rome: Cremonese, 1980.

—. "A Tre Secoli dal Calcolo: La Questione delle Origini." *Bollettino Unione Matematica Italiana* 6 (1984): 1–55.

—. "La *Géométrie* di Descartes tra Numeri e Grandezze." In *Descartes, il Metodo e i Saggi: Atti per il 350º Anniversario della Pubblicazione del Discours de la Méthode e degli Essais*, edited by Giulia Belgioioso, Guido Cimino, Pierre Costabel, and Giovanni Papuli, pp. 419–39. Rome: Istituto dell'Enciclopedia Italiana, 1990.

Glanvill, Joseph. *Scepsis Scientifica: Or, Confest Ignorance, the Way to Science, in an Essay of the Vanity of Dogmatizing, and Confident Opinion, With a Reply to the Exceptions of the Learned Thomas Albius*. London: E. Cotes, for H. Eversden, 1665.

Gregory, David. *Exercitatio Geometrica de Dimensione Figurarum: Sive Specimen Methodi Generalis*. Edinburgh: Ex Officina Typographica Jacobi Kniblo, Josuae Solingensis et Johannis Colmari, 1684.

Gregory, James. *Tercentenary Memorial Volume: Containing His Correspondence with John Collins and His Hitherto Unpublished Mathematical Manuscripts, Together with Addresses and Essays Communicated to the Royal Society of Edinburgh, July 4, 1938*. Edited by Herbert W. Turnbull. London: G. Bell, for the Royal Society of Edinburgh, 1939.

Grosholz, Emily R. *Cartesian Method and the Problem of Reduction*. Oxford: Clarendon Press, 1991.

Gua de Malves, Jean Paul. *Usages de l'Analyse de Descartes: Pour Découvrir, Sans le Secours du Calcul Differentiel, les Proprietés, ou Affections Principales des Lignes Géométriques de Tous les Ordres*. Paris: Chez Briasson et Piget, 1740.

Guerlac, Henry. "Newton and the Method of Analysis." In *Dictionary of the History of Ideas*, edited by Philip P. Wiener, vol. 3, pp. 378–91. New York: Scribner, 1973.

Guicciardini, Niccolò. *The Development of Newtonian Calculus in Britain, 1700–1800*. Cambridge: Cambridge University Press, 1989.

—. *Reading the Principia: The Debate on Newton's Mathematical Methods for Natural Philosophy from 1687 to 1736*. Cambridge: Cambridge University Press, 1999.

Hall, A. Rupert. *Philosophers at War: The Quarrel Between Newton and Leibniz*. Cambridge: Cambridge University Press, 1980.

—. *Isaac Newton: Adventurer in Thought*. Oxford: Blackwell, 1992.

—. "Newton versus Leibniz: From Geometry to Metaphysics." In *The Cambridge Companion to Newton*, edited by I. Bernard Cohen and George E. Smith, pp. 431–54. Cambridge: Cambridge University Press, 2002.

Harper, William L. "Newton's Argument for Universal Gravitation." In *The Cambridge Companion to Newton*, edited by I. Bernard Cohen and George E. Smith, pp. 174–201. Cambridge: Cambridge University Press, 2002.

Harrison, John. *The Library of Isaac Newton*. Cambridge: Cambridge University Press, 1978.

Hayes, Charles. *A Treatise of Fluxions: Or, an Introduction to Mathematical Philosophy.* London: E. Midwinter, for D. Midwinter and T. Leigh, 1704.

Herivel, John. "Newton's Discovery of the Law of Centrifugal Force." *Isis* 51 (1960): 546–53.

—. *The Background to Newton's Principia: A Study of Newton's Dynamical Researches in the Years 1664–1684.* Oxford: Clarendon Press, 1965.

Hill, Katherine. "Neither Ancient nor Modern: Wallis and Barrow on the Composition of Continua: I. Mathematical Styles and the Composition of Continua." *Notes and Records of the Royal Society of London* 50 (1996): 165–78.

Hiscock, Walter George. *David Gregory, Isaac Newton and Their Circle: Extracts from David Gregory's Memoranda 1677–1708.* Oxford: for the Editor, 1937.

Hobbes, Thomas. *Six Lessons to the Professors of the Mathematiques, One of Geometry, the Other of Astronomy: In the Chaires Set Up by the Noble and Learned Sir Henrey Savile, in the University of Oxford.* London: Andrew Crooke, 1656.

—. *De Principiis & Ratiocinatione Geometrarum: Ubi Ostenditur Incertitudinem Falsitatemque Non Minorem Inesse Scriptis Eorum, Quam Scriptis Physicorum & Ethicorum, Contra Fastum Professorum Geometriae.* London: Andrew Crooke, 1666.

—. *The English Works of Thomas Hobbes of Malmesbury.* Edited by William Molesworth. 11 vols. London: Bohn, 1839–45. Reprinted London: Routledge/Thoemmes Press, 1994.

—. *Opera Philosophica Quae Latine Scripsit Omnia: In Unum Corpus Nunc Primum Collecta.* Edited by William Molesworth. 5 vols. London: Bohn, 1839–45.

Hofmann, Joseph E. *Leibniz in Paris 1672–1676: His Growth to Mathematical Maturity.* Cambridge: Cambridge University Press, 1974.

Hooke, Robert. *Micrographia: Or, Some Physiological Descriptions of Minute Bodies Made by Magnifying Glasses, With Observations and Inquiries Thereupon.* London: J. Martyn and J. Allestry, 1665.

Horrocks, Jeremiah. *Opera Posthuma.* London: W. Godbid, for J. Martyn, 1673.

Hunter, Michael. "Hooke the Natural Philosopher." In *London's Leonardo: The Life and Work of Robert Hooke*, edited by Jim Bennett, Michael Cooper, Michael Hunter, and Lisa Jardine, pp. 105–62. Oxford: Oxford University Press, 2003.

Huygens, Christiaan. *Horologium Oscillatorium: Sive, De Motu Pendulorum ad Horologia Aptato Demonstrationes Geometricae.* Paris: F. Muguet, 1673.

—. *Oeuvres Complètes.* 22 vols. The Hague: M. Nijhoff, 1888–1950.

Ihmig, Karl-Norbert. "Newton's Program of Mathematizing Nature." In *Activity and Sign: Grounding Mathematics Education*, edited by Michael H. G. Hoffmann, Johannes Lenhard, and Falk Seeger, pp. 241–61. New York: Springer, 2005.

Iliffe, Robert. "'A Connected System'? The Snare of a Beautiful Hand and the Unity of Newton's Archive." In *Archives of the Scientific Revolution: The Formation and Exchange of Ideas in Seventeenth-century Europe*, edited by Michael Hunter, pp. 137–57. Woodbridge, U.K.: Boydell Press, 1998.

Israel, Giorgio. "Dalle *Regulae* alla *Géométrie*." In *Descartes, il Metodo e i Saggi: Atti per il 350º Anniversario della Pubblicazione del Discours de la Méthode e degli Essais*, edited by Giulia Belgioioso, Guido Cimino, Pierre Costabel, and Giovanni Papuli, pp. 441–74. Rome: Istituto dell'Enciclopedia Italiana, 1990.

Janiak, Andrew. "Newton and the Reality of Force." *Journal of the History of Philosophy* 45 (2007): 127–47.

—. *Newton as Philosopher*. Cambridge: Cambridge University Press, 2008.

Jardine, Nicholas. "Keeping Order in the School of Padua: Jacopo Zabarella and Francesco Piccolomini on the Offices of Philosophy." In *Method and Order in Renaissance Philosophy of Nature: The Aristotle Commentary Tradition*, edited by Daniel A. Di Liscia, Eckhard Kessler, and Charlotte Methuen, pp. 183–209. Aldershot, U.K.: Ashgate, 1997.

Jesseph, Douglas M. *Squaring the Circle: The War Between Hobbes and Wallis*. Chicago: University of Chicago Press, 1999.

Jullien, Vincent. *Descartes: La Géométrie de 1637*. Paris: Presses Universitaires de France, 1996.

Keill, John. "Epistola ad Clarissimum Virum Edmundum Hallejum Geometriae Professorem Savilianum: De Legibus Virium Centripetarum." *Philosophical Transactions* 26 (1708): 174–88. The volume for 1708 issued in 1710.

Kersey, John. *The Elements of That Mathematical Art Commonly Called Algebra, Expounded in Four Books*. London: W. Godbid, for T. Passinger and B. Hurlock, 1673–74.

Kinckhuysen, Gerard. *Algebra, Ofte Stel-Konst*. Haarlem: Passchier van Wesbusch, 1661.

Kneebone, George T. and John G. Semple. *Algebraic Projective Geometry*. Oxford: Oxford University Press, 1998.

Knobloch, Eberhard. "Leibniz et Son Manuscrit Inédité sur la Quadrature des Sections Coniques." In *The Leibniz Renaissance*, pp. 127–51. Florence: Olschki, 1989.

Knoespel, Kenneth J. "Interpretive Strategies in Newton's *Theologiae Gentilis Origines Philosophicae*." In *Newton and Religion: Context, Nature and Influence*, edited by James E. Force and Richard H. Popkin, pp. 179–202. Dordrecht: Kluwer, 1999.

Knorr, Wilbur R. *The Ancient Tradition of Geometric Problems*. New York: Dover, 1993.

Kollerstrom, Nicholas. "Thomas Simpson and 'Newton's Method of Approximation': An Enduring Myth." *British Journal for the History of Science* 25 (1992): 347–54.

—. "The Path of Halley's Comet, and Newton's Late Apprehension of the Law of Gravity." *Annals of Science* 56 (1999): 331–56.

—. *Newton's Forgotten Lunar Theory: His Contribution to the Quest for Longitude*. Santa Fe, N.M.: Green Lion Press, 2000.

La Hire, Philippe de. *Nouvelle Methode en Geometrie Pour les Sections des Superficies Coniques et Cylindriques, Qui Ont Pour Bases des Cercles, ou des Paraboles, des Elipses, & des Hyperboles*. Paris: Chez l'Autheur et T. Moette, 1673.

Le Seur, Thomas, and François Jacquier. *Elemens du Calcul Intégral*. 2 vols. Parma: Chez les Héritiers Monti, 1768.

Leibniz, Gottfried W. "Nova Methodus pro Maximis et Minimis, Itemque Tangentibus, Quae nec Fractas, nec Irrationales Quantitates Moratur, et Singulare pro Illis Calculi Genus." *Acta Eruditorum* (1684): 467–73.

—. "De Geometria Recondita et Analysi Indivisibilium atque Infinitorum." *Acta Eruditorum* (1686): 292–300.

—. "Analysis per Quantitatum, Series, Fluxiones ac Differentias Cum Enumeratione Linearum Tertii Ordinis." *Acta Eruditorum* (1712): 74–7. Leibniz's anonymous review of Newton, *Analysis per Quantitatum* (1711).

—. *Leibnizens Mathematische Schriften*. Edited by Carl Immanuel Gerhardt. 7 vols. Berlin: Asher, 1849–63. Reprinted Hildesheim, Germany: Georg Olms, 1971.

—. *De Quadratura Arithmetica Circuli Ellipseos et Hyperbolae Cujus Corollarium est Trigonometria sine Tabulis*. Edited by Eberhard Knobloch. Göttingen, Germany: Vandenhoeck and Ruprecht, 1993.

—. *The Labyrinth of the Continuum: Writings on the Continuum Problem, 1672–1686*. Translated, edited, and with an introduction by Richard T. W. Arthur. New Haven: Yale University Press, 2001.

L'Hospital, Guillaume F. A. de. *Analyse des Infiniment Petits: Pour l'Intelligence des Lignes Courbes*. Paris: Imprimerie Royale, 1696.

Love, Harold. *Scribal Publication in Seventeenth-Century England*. Oxford: Clarendon Press, 1993.

Maclaurin, Colin. *Geometria Organica: Sive Descriptio Linearum Curvarum Universalis*. London: W. and J. Innys, 1720.

—. *An Account of Sir Isaac Newton's Philosophical Discoveries*. London: Printed for the Author's children, and sold by A. Millar, J. Nourse, G. Hamilton, J. Balfour, and A. Kincaid at Edinburgh, J. Barry at Glasgow, and J. Smith at Dublin, 1748. Edition cited: 3d ed., 1775.

Mahoney, Michael S. *The Mathematical Career of Pierre de Fermat (1601–1665)*. Princeton: Princeton University Press, 1973. 2d rev. ed., 1994.

—. "Barrow's Mathematics: Between Ancients and Moderns." In *Before Newton: The Life and Times of Isaac Barrow*, edited by Mordechai Feingold, pp. 179–249. Cambridge: Cambridge University Press, 1990.

—. "The Mathematical Realm of Nature." In *The Cambridge History of Seventeenth-century Philosophy*, edited by Michael Ayers and Daniel Garber, pp. 702–55. Cambridge: Cambridge University Press, 1998.

Maierù, Luigi. *John Wallis: Una Vita Per un Progetto*. Soveria Mannelli, Italy: Rubbettino, 2007.

Malet, Antoni. *From Indivisibles to Infinitesimals: Studies on Seventeenth-Century Mathematizations of Infinitely Small Quantities*. Bellaterra, Spain: Universitat Autònoma de Barcelona, 1996.

—. "Barrow, Wallis, and the Remaking of Seventeenth-Century Indivisibles." *Centaurus* 39 (1997): 67–92.

—. “Isaac Barrow on the Mathematization of Nature: Theological Voluntarism and the Rise of Geometrical Optics.” *Journal of the History of Ideas* 58 (1997): 265–87.

Mamiani, Maurizio. “La Rivoluzione Incompiuta: I Confini tra Matematica e Fisica da Newton a d'Alembert.” In *L'Età dei Lumi: Saggi sulla Cultura Settecentesca*, edited by Antonio Santucci, pp. 39–48. Bologna: Il Mulino, 1998.

Mancosu, Paolo. *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century*. Oxford: Oxford University Press, 1996.

Mandelbrote, Scott. “The Heterodox Career of Nicolas Fatio de Duillier.” In *Heterodoxy in Early Modern Science and Religion*, edited by John Brooke and Ian MacLean, pp. 263–96. Oxford: Oxford University Press, 2005.

McGuire, James E. “Newton's ‘Principles of Philosophy’: An intended Preface for the 1704 *Opticks* and a Related Draft Fragment.” *British Journal for the History of Science* 5 (1970): 178–86.

—. “Existence, Actuality and Necessity: Newton on Space and Time.” *Annals of Science* 25 (1978): 463–508.

McMullin, Ernan. *Newton on Matter and Activity*. Notre Dame, Ind.: University of Notre Dame Press, 1978.

Mercator, Nicolaus (Kauffman, Niklaus). *Logarithmo-technia: Sive Methodus Construendi Logarithmos Nova, Accurata, & Facilis, Scripto Antehàc Communicata, Anno sc. 1667 Nonis Augusti, Cui Nunc Accedit, Vera Quadratura Hyperbolae, & Inventio Summae Logarithmorum*. London: by W. Godbid, for M. Pitt, 1668.

Miller, Maximilian. “Newton, Aufzählung der Linien Dritter Ordnung.” *Wissenschaftliche Zeitschrift der Hochschule für Verkehrswesen Dresden* 1 (1953): 5–32.

Milne, John J. “Newton's Contribution to the Geometry of Conics.” In *Isaac Newton, 1642–1727: A Memorial Volume Edited for the Mathematical Association*, edited by William J. Greenstreet, pp. 96–114. London: G. Bell, 1927.

Molland, Andrew George. “Shifting the Foundations: Descartes' Transformation of Ancient Geometry.” *Historia Mathematica* 3 (1976): 21–49.

Montucla, Jean E. *Histoire des Mathématiques*. 4 vols. Paris: Henri Agasse, 1799–1802. Volumes 3 and 4 were edited by Joseph Jérôme L. de Lalande, assisted by a number of other scholars.

Murdoch, Patrick. *Neutoni Genesis Curvarum per Umbras: Seu Perspectivae Universalis Elementa, Exemplis Coni Sectionum et Linearum Tertii Ordinis Illustrata*. London: A. Millar, 1746.

Nauenberg, Michael. “Hooke, Orbital Motion, and Newton's Principia.” *American Journal of Physics* 62 (1994): 331–50.

—. “Newton's Early Computational Method for Dynamics.” *Archive for History of Exact Sciences* 46 (1994): 221–52.

—. “Newton's Perturbation Methods for the Three-Body Problem and Their Application to Lunar Motion.” In *Isaac Newton's Natural Philosophy*, edited by Jed Z. Buchwald and I. Bernard Cohen, pp. 189–224. Cambridge, Mass.: MIT Press, 2001.

—. "Kepler's Area Law in the *Principia*: Filling in Some Details in Newton's Proof of Proposition 1." *Historia Mathematica* 30 (2003): 441–56.

—. "Robert Hooke's Seminal Contributions to Orbital Dynamics." *Physics in Perspective* 7 (2005): 4–34.

Neal, Katherine. *From Discrete to Continuous: The Broadening of Number Concepts in Early Modern England*. Dordrecht: Kluwer, 2002.

Newman, William R., and Lawrence M. Principe. "Alchemy vs. Chemistry: The Etymological Origins of a Historiographic Mistake." *Early Science and Medicine* 3 (1998): 32–65.

Newton, Isaac. *Philosophiae Naturalis Principia Mathematica*. London: J. Streater, 1687.

—. "A Letter of Mr. Isaac Newton, Professor of the Mathematicks in the University of Cambridge: Containing His New Theory about Light and Colors." *Philosophical Transactions* 80 (February 1671/2): 3075–87.

—. *A New and Most Accurate Theory of the Moon's Motion . . . Published in Latin by D. Gregory in his Excellent Astronomy*. London: A. Baldwin, 1702.

—. *Opticks: Or, A Treatise of the Reflexions, Refractions, Inflexions and Colours of Light, Also Two Treatises of the Species and Magnitude of Curvilinear Figures*. London: for S. Smith and B. Walford, 1704.

—. *Optice: Sive de Reflexionibus, Refractionibus, Inflexionibus & Coloribus Lucis Libri Tres, Authore Isaaco Newton, Latine Reddidit Samuel Clarke, Accedunt Tractatus Duo Eiusdem Authoris de Speciebus & Magnitudine Figurarum Curvilinearum, Latine Scripti*. Translated by Samuel Clarke. London: for S. Smith and B. Walford, 1706.

—. *Arithmetica Universalis: Sive de Compositione et Resolutione Arithmetica Liber*. Edited by William Whiston. Cambridge: Cambridge University Press, 1707.

—. *Analysis per Quantitatum, Series, Fluxiones, ac Differentias: Cum Enumeratione Linearum Tertii Ordinis*. Edited by William Jones. London: Pearson, 1711.

—. *Philosophiae Naturalis Principia Mathematica*. Edited by Roger Cotes. Cambridge: Cornelius Crownfield, 1713.

—. "An Account of the Book Entituled Commercium Epistolicum Collinii & Aliorum de Analysi Promota; Published by Order of the Royal Society, in Relation to the Dispute between Mr. Leibnitz and Dr. Keill, About the Right Invention of the Method of Fluxions, by Some Call'd the Differential Method." *Philosophical Transactions* 29, no. 342 (1715): 173–224. Facsimile in Hall, *Philosophers at War* (1980), pp. 263–314.

—. *Universal Arithmetick: Or, a Treatise of Arithmetical Composition and Resolution*. Translated by Joseph Raphson, revised by Samuel Cunn. London: for J. Senex, W. Taylor, T. Warner, and J. Osborn, 1720.

—. *Philosophiae Naturalis Principia Mathematica*. Edited by Henry Pemberton. London: W. and J. Innys, 1726.

—. *Optical Lectures: Read in the Publick Schools of the University of Cambridge, Anno Domini, 1669 . . . Never Before Printed, Translated into English out of the Original Latin*. London: for F. Fayram, 1728.

—. *Lectiones Opticae: Annis MDCLXIX, MDCLXX & MDCLXXI, In Scholis Publicis Habitae, et Nunc Primum ex MSS. in Lucem Editae*. London: W. Innys, 1729.

—. *The Mathematical Principles of Natural Philosophy, by Sir Isaac Newton, Translated into English By Andrew Motte: To Which are Added, The Laws of the Moon's Motion, According to Gravity, By John Machin*. Translated by Benjamin Motte. 2 vols. London: for B. Motte, 1729.

—. *The Method of Fluxions and Infinite Series: With its Application to the Geometry of Curve-lines, By the Inventor Sir Isaac Newton, Kt. Late President of the Royal Society, Translated From the Author's Latin Original Not Yet Made Publick, To Which is Subjoin'd, a Perpetual Comment upon the Whole Work, Consisting of Annotations, Illustrations, and Supplements, In Order to Make This Treatise a Compleat Institution For the Use of Learners*. Translated and annotated by John Colson. London: H. Woodfall for J. Nourse, 1736.

—. *La Methode [sic] des Fluxions, et des Suites Infinies*. Translated by Georges-Louis Leclerc de Buffon. Paris: De Bure, 1740.

—. *Opuscula Mathematica, Philosophica et Philologica, Collegit Partimque Latine Vertit ac Recensuit Joh. Castillioneus ... Accessit Commentariolus de Vita Auctoris*. Edited by Castiglione (Giovan Francesco Salvemini). 3 vols. Lausanne: Bousquet, 1744.

—. *Two Treatises of the Quadrature of Curves, and Analysis by Equations of an Infinite Number of Terms, Explained: Containing the Treatises Themselves, Translated into English, With a Large Commentary, in Which the Demonstrations Are Supplied Where Wanting, the Doctrine Illustrated, and the Whole Accomodated to the Capacities of Beginners, for Whom It Is Chiefly Designed*. Translation and commentary by John Stewart. London: J. Bettenham, for Society for the Encouragement of Learning, sold by J. Nourse and John Whiston, 1745.

—. *Philosophiae Naturalis Principia Mathematica, Auctore Isaaco Newtono, Eq. Aurato: Perpetuis Commentariis Illustrata, Communi Studio PP. Thomae Le Seur & Francisci Jacquier Ex Gallicana Minimorum Familia, Matheseos Professorum*. Edited by Thomas Le Seur and François Jacquier. 3 vols. Geneva: Typis Barrillot, 1739–42.

—. *Opera Quae Exstant Omnia*. Edited by Samuel Horsley. 5 vols. London: J. Nichols, 1779–85.

—. *Enumeration of Lines of the Third Order: Generation of Curves by Shadows, Organic Description of Curves, and Construction of Equations by Curves*. Translation and commentary by Christopher R. M. Talbot. London: Bohn, 1861.

—. *The Correspondence of Isaac Newton*. Edited by Herbert W. Turnbull, John F. Scott, A. Rupert Hall, and Laura Tilling. 7 vols. Cambridge: Cambridge University Press, 1959–1977.

—. *Unpublished Scientific Papers of Isaac Newton: A Selection From the Portsmouth Collection in the University Library Cambridge*. Edited by A. Rupert Hall and Mary Boas Hall. Cambridge: Cambridge University Press, 1962.

—. "De Gravitatione et Aequipondio Fluidorum." In *Unpublished Scientific Papers of Isaac Newton: A Selection From the Portsmouth Collection in the University Library, Cambridge*, edited by A. Rupert Hall and Mary Boas Hall, pp. 89–121. Cambridge: Cambridge University Press, 1962.

—. *The Mathematical Works of Isaac Newton.* Edited by Derek T. Whiteside. 2 vols. New York: Johnson Reprint Corp., 1964–67.

—. *The Mathematical Papers of Isaac Newton.* Edited by Derek T. Whiteside. 8 vols. Cambridge: Cambridge University Press, 1967–81.

—. *Philosophiae Naturalis Principia Mathematica, The Third Edition (1726) With Variant Readings.* Edited by Alexandre Koyré and I. Bernard Cohen, assisted by Anne Whitman. Cambridge: Cambridge University Press, 1972.

—. *The Unpublished First Version of Isaac Newton's Cambridge Lectures on Optics, 1670–1672: A Facsimile of the Autograph, Now Cambridge University Library MS Add. 4002.* Introduction by Derek T. Whiteside. Cambridge: Cambridge University Press, 1973.

—. *Certain Philosophical Questions: Newton's Trinity Notebook.* Edited by James E. McGuire and Martin Tamny. Cambridge: Cambridge University Press, 1983.

—. *The Optical Papers of Isaac Newton. Vol.1. The Optical Lectures 1670–1672.* Edited by Alan E. Shapiro. Cambridge: Cambridge University Press, 1984.

—. *The Preliminary Manuscripts for Isaac Newton's 1687 Principia, 1684–1686.* Edited by Derek T. Whiteside. Cambridge: Cambridge University Press, 1989.

—. *The Principia: Mathematical Principles of Natural Philosophy ... Preceded by a Guide to Newton's Principia by I. Bernard Cohen.* Translated by I. Bernard Cohen and Anne Whitman, assisted by Julia Budenz. Berkeley: University of California Press, 1999.

—. *Análisis de Cantidades Mediante Series, Fluxiones y Differencias: Con una Enumeración de las Líneas de Tercer Orden.* Translation by J. L. Arantegui Tamayo, commentary by A. J. Durán Guardeño, with essays by J. M. Sánchez Ron, A. J. Durán Guardeño, and J. Echeverría. Madrid: Real Sociedad Matemática Española, 2003.

—. *Philosophical Writings.* Edited by Andrew Janiak. Cambridge: Cambridge University Press, 2004.

Omerique, Antonio Hugo de. *Analysis Geometrica: Sive Nova, et Vera Methodus Resolvendi Tam Problemata Geometrica, Quam Arithmeticas Quaestiones, Pars Prima De Planis.* Cádiz: Typis Christophori de Requena, 1698.

Oughtred, William. *Clavis Mathematicae Denuo Limata, Sive Potius Fabricata.* Oxford: Lichfield, 1652. 1st ed. 1631.

Ozanam, Jacques. *Dictionaire Mathématique: Ou Idée Générale des Mathématiques.* Amsterdam: Huguetan, 1691. 1st ed. Paris, 1690.

—. *La Perspective Theorique et Pratique: Ou l'On Enseigne la Maniere de Mettre Toutes Sortes d'Objets en Perspective, & d'en Representer les Ombres Causées Par le Soleil ou Par une Petite Lumiere.* Paris: C. Jombert, 1711.

Panza, Marco. *Newton et les Origines de l'Analyse: 1664–1666.* Paris: Blanchard, 2005.

Pappus. *Mathematicae Collectiones à Federico Commandino Urbinate in Latinum Conversae, et Commentariis Illustratae.* Pesaro, Italy: Girolamo Concordia, 1588.

—. *Collectionis Quae Supersunt.* Translated and edited by Friedrich Hultsch. 3 vols. Berlin: Weidmannschen Buchhandlung, 1876–78.

—. *Book 7 of the Collection.* Edited by Alexander Jones. 2 vols. New York: Springer, 1986.

Parshall, Karen H. *James Joseph Sylvester: Jewish Mathematician in a Victorian World.* Baltimore: Johns Hopkins University Press, 2006.

Pelseneer, Jean. "Une Opinion Inédite de Newton sur 'l'Analyse des Anciens' à Propos de l'*Analysis Geometrica* de Hugo de Omerique." *Isis* 14 (1930): 155–65.

Pemberton, Henry. *A View of Sir Isaac Newton's Philosophy.* London: S. Palmer, 1728.

Pesic, Peter. "The Validity of Newton's Lemma 28." *Historia Mathematica* 28 (2001): 215–19.

Piccolomini, Alessandro. *In Mechanicas Quaestiones Aristotelis Paraphrasis Paulo Quidem Plenior: Eiusdem Commentarium de Certitudine Mathematicarum Disciplinarum, In Quo, de Resolutione, Diffinitione, & Demonstratione, Necnon de Materia, et in Fine Logicae Facultatis, Quamplura Continentur ad Rem Ipsam, tum Mathematicam, tum Logicam, Maxime Pertinentia.* Roma: Apud A. Bladum, 1547.

Pitcairn, Archibald. *Solutio Problematis de Historicis, seu Inventoribus.* Edinburgh: J. Reid, 1688.

Popkin, Richard H. *The History of Scepticism: From Savonarola to Bayle.* Oxford: Oxford University Press, 2003.

Pourciau, Bruce. "On Newton's Proof that Inverse-Square Orbits Must Be Conics." *Annals of Science* 48 (1991): 159–72.

—. "Newton's Solution of the One-Body Problem." *Archive for History of Exact Sciences* 44 (1992): 125–46.

—. "The Preliminary Mathematical Lemmas of Newton's *Principia*." *Archive for History of Exact Sciences* 52 (1998): 279–95.

—. "The Integrability of Ovals: Newton's Lemma 28 and Its Counterexamples." *Archive for History of Exact Sciences* 55 (2001): 479–99.

—. "Newton's Argument for Proposition 1 of the *Principia*." *Archive for History of Exact Sciences* 57 (2003): 267–311.

—. "Proposition II (Book I) of Newton's *Principia*." *Archive for History of Exact Sciences* 63 (2009): 129–167.

Principe, Lawrence M. "The Alchemies of Robert Boyle and Isaac Newton: Alternate Approaches and Divergent Deployments." In *Rethinking the Scientific Revolution*, edited by Margaret J. Osler, pp. 201–20. Cambridge: Cambridge University Press, 2000.

Puiseux, Victor-Alexandre. "Recherches sur les Fonctions Algébriques." *Journal de Mathématiques Pures et Appliquées* 15 (1850): 365–480.

Pycior, Helena M. "Mathematics and Philosophy: Wallis, Hobbes, Barrow, and Berkeley." *Journal of the History of Ideas* 48 (1987): 265–86.

—. *Symbols, Impossible Numbers, and Geometric Entanglements: British Algebra Through the Commentaries on Newton's Universal Arithmetick.* Cambridge: Cambridge University Press, 1997.

Rahn, Johann H., and John Pell. *An Introduction to Algebra*, translated by Thomas Brancker, prefaced and written with contributions of John Pell. London: for M. Pitt, 1668.

Raphson, Joseph. *The History of Fluxions: Shewing in a Compendious Manner the First Rise Of, and Various Improvements Made in That Incomparable Method*. London: Pearson, for R. Mount, 1715. 2d ed. 1717. In 1715 there was also a Latin edition.

Rashed, Roshdi. "La *Géométrie* de Descartes et la Distinction Entre Courbes Géométriques et Courbes Mécaniques." In *Descartes et le Moyen-Âge*, edited by Joël Biard and Roshdi Rashed, pp. 11–26. Paris: Vrin, 1997.

Riccati, Vincenzo, and Girolamo Saladini. *Institutiones Analyticae*. 2 vols. Bologna: Ex Typographia Sancti Thomae Aquinatis, 1765–67.

Rigaud, Stephen P. *Historical Essay on the First Publication of Sir Isaac Newton's Principia*. Oxford: Oxford University Press, 1838.

Rigaud, Stephen P., and Stephen J. Rigaud. *Correspondence of Scientific Men of the Seventeenth Century*. 2 vols. Oxford: Oxford University Press, 1841.

Robins, Benjamin. *Mathematical Tracts*. Edited by James Wilson. London: J. Nourse, 1761.

Rouse Ball, William W. "On Newton's Classification of Cubic Curves." *Proceedings of the London Mathematical Society* 22 (1891): 104–43.

Ruffner, James A. "Newton's Propositions on Comets: Steps in Transition, 1981–84." *Archive for History of Exact Sciences* 54 (2000): 259–77.

Sageng, Erik. Colin MacLaurin and the Foundations of the Method of Fluxions. Ph.D. diss., Princeton University, 1989.

Saint Vincent, Grégoire de. *Opus Geometricum Quadraturae Circuli et Sectionum Coni Decem Libris Comprehensum*. Antwerp: Apud Ioannem et Iacobum Meursios, 1647.

Salmon, George. *A Treatise on Conic Sections: Containing an Account of Some of the Most Important Modern Algebraic and Geometric Methods*. 10th ed. London: Longmans, Green, 1896.

Salusbury, Thomas. *Mathematical Collections and Translations in Two Tomes (Englished from the Originall Latine and Italian)*. 2 vols. London: W. Leybourn, 1661–65.

Sarasa, Alphonse Antonio de. *Solutio Problematis a R. P. Marino Mersenno Minimo Propositi*. Antwerp: Apud Ioannem et Iacobum Meursios, 1649.

Sasaki, Chikara. "The Acceptance of the Theory of Proportion in the Sixteenth and Seventeenth Centuries: Barrow's Reaction to the Analytic Mathematics." *Historia Scientiarum* 29 (1985): 83–116.

Schneider, Ivo. "Direct and Indirect Influences of Jakob Bernoulli's *Ars Conjectandi* in 18th Century Great Britain." *Electronic Journal for History of Probability and Statistics* 2, no. 1 (June 2006): 1–17.

Schooten, Frans van. *Exercitationum Mathematicarum Libri Quinque: I. Propositionum Arithmeticarum et Geometricarum Centuria, II. Constructio Problematum Simplicium Geometricorum, III. Apollonii Pergaei Loca Plana Restituta, IV. Organica Conicarum Sec-*

tionum in Plano Descriptio, V. Sectiones Miscellaneae Triginta, Quibus Accedit Christiani Hugenii Tractatus De Ratiociniis in Aleae Ludo. Leiden: Ex Officina Johannis Elsevirii, 1657.

Scott, John F. "John Wallis as a Historian of Mathematics." *Annals of Science* 1 (1936): 335–57.

—. *The Mathematical Work of John Wallis, D.D., F.R.S. (1616–1703).* 2d ed. Foreword by E. N. da C. Andrade. New York: Chelsea Publishing, 1981.

Scriba, Christoph J. "John Wallis." In *Dictionary of Scientific Biography*, edited by Charles C. Gillispie, vol. 14, pp. 146–55. New York: Scribner, 1976.

Sellés, Manuel A. "Infinitesimals in the Foundations of Newton's Mechanics." *Historia Mathematica* 33 (2006): 210–23.

Sepkoski, David. "Nominalism and Constructivism in Seventeenth-Century Mathematical Philosophy." *Historia Mathematica*, 32 (2005): 33–59.

—. *Nominalism and Constructivism in Seventeenth-Century Mathematical Philosophy.* London: Routledge, 2007.

Serfati, Michel. "Les Compas Cartésiens." *Archives de Philosophie* 56 (1993): 197–230.

—. *La Révolution Symbolique: La Constitution de l'Écriture Symbolique Mathématique.* Paris: Petra, 2005.

Shapin, Steven. "Robert Boyle and Mathematics: Reality, Representation, and Experimental Practice." *Science in Context* 2 (1988): 23–58.

Shapiro, Alan E. *Fits, Passions, and Paroxysms: Physics, Method, and Chemistry and Newton's Theories of Colored Bodies and Fits of Easy Reflection.* Cambridge: Cambridge University Press, 1993.

—. "Newton's 'Experimental Philosophy'." *Early Science and Medicine* 9 (2004): 185–217.

Shapiro, Barbara J. *Probability and Certainty in Seventeenth-Century England: A Study of the Relationships Between Natural Science, Religion, History, Law and Literature.* Princeton: Princeton University Press, 1983.

Shkolenok, Galina A. "Geometrical Constructions Equivalent to Non-Linear Algebraic Transformations of the Plane in Newton's Early Papers." *Archive for History of Exact Sciences* 9 (1972): 22–44.

Simson, Robert. *Opera Quaedam Reliqua ... Nunc Primum Post Auctoris Mortem in Lucem Edita.* Glasgow: R. and A. Foulis, 1776.

Smith, George. "The Methodology of the *Principia*." In *The Cambridge Companion to Newton*, edited by I. Bernard Cohen and George E. Smith, pp. 138–73. Cambridge: Cambridge University Press, 2002.

Snobelen, Stephen. "'The True Frame of Nature': Isaac Newton, Heresy and the Reformation of Natural Philosophy." In *Heterodoxy in Early Modern Science and Religion*, edited by John Brooke and Ian MacLean, pp. 223-62. Oxford: Oxford University Press, 2005.

Stedall, Jacqueline A. "'Of Our Own Nation': John Wallis's Account of Mathematical Learning in Medieval England." *Historia Mathematica* 28 (2001): 73–122.

—. *'A Discourse Concerning Algebra': English Algebra to 1685*. Oxford: Oxford University Press, 2002.

—. "John Wallis, *Arithmetica Infinitorum*." In *Landmark Writings in Western Mathematics 1640–1940*. Edited by Ivor Grattan-Guinness, pp. 23–32. Amsterdam, Oxford: Elsevier, 2005.

—. "Newton's Algebra." In *The Reception of Isaac Newton in Europe*. Edited by Scott Mandelbrote and Helmut Pulte. London and New York: Continuum Press, forthcoming.

Stewart, Ian. "Mathematics as Philosophy: Proclus and Barrow." *Dionysius* 18 (2000): 151–81.

Stirling, James. *Lineae Tertii Ordinis Neutonianae: Sive Illustratio Tractatus D. Neutoni de Enumeratione Linearum Tertii Ordinis, Cui Subjungitur, Solutio Trium Problematum*. Oxford: E Theatro Sheldoniano, 1717.

Sylvester, James Joseph. "Algebraical Researches, Containing a Disquisition on Newton's Rule for the Discovery of Imaginary Roots." *Philosophical Transactions* 154 (1864): 579–666.

—. "On an Elementary Proof and Generalisation of Sir Isaac Newton's Hitherto Undemonstrated Rule for the Discovery of Imaginary Roots." *Proceedings of the London Mathematical Society* 1 (1865): 1–16.

Torricelli, Evangelista. *Opera Geometrica*. Florence: A. Masse and L. de Landis, 1644.

Turnbull, Herbert W. *The Mathematical Discoveries of Newton*. London: Blackie, 1945.

Valluri, Sree Ram, Curtis Wilson, and William L. Harper. "Newton's Apsidal Precession Theorem and Eccentric Orbits." *Journal for the History of Astronomy* 28 (1997): 13–27.

Vermij, Rienk H., and Jan A. van Maanen. "An Unpublished Autograph by Christiaan Huygens: His Letter to David Gregory of 19 January 1694." *Annals of Science* 49 (1992): 507–23.

Viète, François. *In Artem Analiticem Isagoge: Seorsim Excussa ab Opere Restitutae Mathematicae Analyseos, seu Algebra Nova*. Tours: Apud J. Mettayer, 1591.

—. *Opera Mathematica: In Unum Volumen Congesta, ac Recognita*. Edited by Frans van Schooten. Leiden: Ex Officina Bonaventurae et Abrahami Elzeviriorum, 1646.

Voss, Gerardus Joannes. *De Universae Mathesios Natura & Constitutione Liber*. Amsterdam: Ex Typographeio Ioannis Blaeu, 1660. Part of *De Quatuor Artibus Popularibus, de Philologia, et Scientiis Mathematicis: Cui Operi Subjungitur Chronologia Mathematicorum, Libri Tres*.

Wallis, John. *Arithmetica Infinitorum: Sive Nova Methodus Inquirendi in Curvilinearum Quadraturam, Aliaque Difficliliora Matheseos Problemata*. Oxford: Lichfield, 1655. Printed by July 1655; published in Wallis, *Operum Mathematicorum, Pars Altera* (1656), pp. 1–199.

—. *Operum Mathematicorum Pars Altera: Qua Continentur, de Angulo Contactus et Semicirculi Disquisitio Geometrica, de Sectionibus Conicis Tractatus, Arithmetica Infinitorum, Sive de Curvilineorum Quadratura ... Eclipseos Solaris Observatio*. Oxford: Lichfield, for O. Pullein, 1656.

—. *Commercium Epistolicum de Quaestionibus Quibusdam Mathematicis Nuper Habitum.* Oxford: Excudebat A. Lichfield, impensis T. Robinson, 1658.

—. *Mechanica: Sive de Motu, Tractacus Geometricus.* London: W. Godbid, for M. Pitt, 1670–1.

—. *A Defense of the Treatise of the Angle of Contact.* London: John Playford, for Richard Davis, 1684.

—. *A Treatise of Algebra, Both Historical and Practical: Shewing the Original, Progress, and Advancement Thereof, From Time to Time, and by What Steps It Hath Attained to the Heighth at Which It Now Is, With Some Additional Treatises, I. Of the Cono-Cuneus ... II. Of Angular Sections, and Other Things Relating Thereunto, and to Trigonometry, III. Of the Angle of Contact ... IV. Of Combinations, Alternations, and Aliquot Parts.* London: John Playford, for Richard Davis, 1685.

—. *Opera Mathematica.* 3 vols. Oxford: E Theatro Sheldoniano, 1693–99.

—. *The Arithmetic of Infinitesimals.* Translation and commentary by Jacqueline A. Stedall. New York: Springer, 2004.

Westfall, Richard S. *Never at Rest: A Biography of Isaac Newton.* Cambridge: Cambridge University Press, 1980.

—. "Newton's Marvelous Years of Discovery and Their Aftermath: Myth Versus Manuscript." *Isis* 71 (1980): 109–21.

Whewell, William. *History of the Inductive Sciences: From the Earliest to the Present Times.* London: J. W. Parker, 1837.

Whiteside, Derek T. "Patterns of Mathematical Thought in the Later Seventeenth Century." *Archive for History of Exact Sciences* 1 (1961): 179–388.

—. "Newton's Discovery of the General Binomial Theorem." *Mathematical Gazette* 45 (1961): 175–80.

—. "Isaac Newton: Birth of a Mathematician." *Notes and Records of the Royal Society of London* 19 (1964): 53–62.

—. "Newton's Early Thoughts on Planetary Motion: A Fresh Look." *The British Journal for the History of Science* 2 (1964): 117–37.

—. "Newton's Marvellous Year: 1666 and All That." *Notes and Records of the Royal Society of London* 21 (1966): 32–41.

—. "Sources and Strengths of Newton's Early Mathematical Thought." *The Texas Quarterly* 10 (1967): 69–85.

—. "Before the Principia: The Maturing of Newton's Thoughts on Dynamical Astronomy, 1664–1684." *Journal for the History of Astronomy* 1 (1970): 5–19.

—. "The Mathematical Principles Underlying Newton's *Principia Mathematica*." *Journal for the History of Astronomy* 1 (1970): 116–38.

—. "Newton's Lunar Theory: From High Hope to Disenchantment." *Vistas in Astronomy* 19 (1976): 317–28.

—. "The Prehistory of the *Principia* from 1664 to 1686." *Notes and Records of the Royal Society of London* 45 (1991): 11–61.

Wilson, Curtis. "Newton on the Moon's Variation and Apsidal Motion: The Need for a Newer 'New Analysis'." In *Isaac Newton's Natural Philosophy*, edited by Jed Z. Buchwald and I. Bernard Cohen, pp. 139–88. Cambridge, Mass.: MIT Press, 2001.

—. "Newton and Celestial Mechanics." In *The Cambridge Companion to Newton*, edited by I. Bernard Cohen and George E. Smith, pp. 202–26. Cambridge: Cambridge University Press, 2002.

Wren, Christopher. *Parentalia: Or, Memoirs of the Family of the Wrens*. Edited by Christopher Wren, Joseph Ames, and Stephen Wren. London: for T. Osborn, etc., 1750.

Yoder, Joella G. *Unrolling Time: Christiaan Huygens and the Mathematization of Nature*. Cambridge: Cambridge University Press, 1988.

Index